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ON THE MAXIMUM TERMS, ORDERS AND TYPES OF THE DERIVATIVES OF AN ENTIRE FUNCTION IN SEVERAL COMPLEX VARIABLES.

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Abstract : Let F be the family of all entire functions in the complex n-space C^n . For f, g ϵ F, the Hadamard product $f * g$ is defined. Certain inequalities involving maximum modulii of the derivatives of the Hadamard product and the Hadamard product of the derivatives of f and g have been obtained. A few relations involving maximum term and the corresponding rank of the derivatives of the above product have also been obtained. Two kinds of orders and types have been considered and a few results involving them have been obtained.

1. **Notations :** We denote complex and real n-space by C^n and R^n respectively and the set of non-negative integers by I, so that I^n denotes the Cartesian product of n-copies of I. We indicate the points $(z_1, \ldots , z_n), (r_1, \ldots, r_n), (m_1, \ldots , m_n)$ etc of C^n or R^n by their corresponding unsuffixed symbols z, r, m, etc. For z, w ϵ C^n and $\alpha \epsilon C$ we define z = w iff

$z_i = w_i, i = i, \ldots, n$

$\alpha z = \alpha z_1, \cdots \alpha z_n)$

$z + w = (z_1 + w_1, \ldots, z_n + w_n)$;

$| z | = \{ | z_1 |^2 + \ldots + | z_n |^2\}^{\frac{1}{2}}$

The positive hyper octant R_+^n is the set $R^n = \{ x : x \epsilon R^n , x_i \geqslant o, i = 1, \ldots, n\}$. For k ϵ R_+^n we set $\| k \| = k_1 + \ldots + k_n$ and for m ϵ I^n, $m ! = m_1 ! \ldots m_n !$.

For any p ϵ I, $\bar{p}$ will stand for the n-tuple (p, ..., p). Also for z ϵ C^n , k ϵ R_+^n, we shall write $z^k = z_1^{k_1} \dots z_n^{k_n}$ ($z_i^o = 1$ even if $z_i = O$).

For any $x, y \in R^n$, we say that

(1) $x \leqslant y$ iff $x_i \leqslant y_i$, $i = 1, \dots, n$

(2) $x < y$ iff $x \leqslant y$ but $x \neq y$

(3) $x << y$ iff $x_i < y_i$, $i = 1, \dots, n$

For an entire function f with domain C^n, $f^{(k)}$ will denote the function

$$\frac{\partial^{\|k\|} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}, \quad k \in I^n.$$

We write for any non-empty complete n-circular domain D [for definition vide 1 § 3.3] with centre at $\bar{O} = (O, \dots, O)$ in C^n. $|D| = \{ r : |z_i| = r_i, z \in D \}$

and $D^+ = \{ r : r \in |D|, \text{ no } r_i = O \}$

For any $r = (r_1, \dots, r_n) \in R^n$ we write $r = \prod_{i=1}^{n} r_i$

2. Let F be the family of all entire functions in C^n represented by a multiple power series of the form

(2.1) $$f(z) = \sum_{\|m\| = o}^{\infty} a_m z^m$$

For f, $g \in F$ we define Hadamard product $f * g$ by

(2.2) $$(f * g)(z) = \sum_{\|m\| = o}^{\infty} a_m b_m z^m, \quad \text{where}$$

(2.3) $$g(z) = \sum_{\|m\| = o}^{\infty} b_m z^m$$

we see

(2.4) $$f^{(k)}(z) = \sum_{\|m\| = o}^{\infty} \left[\prod_{i=1}^{n} \left\{ \prod_{j=o}^{k_i - 1} (m_i - j) \right\} \right] a_m z^{m-k}$$

Evidently $f * g$ belongs to F.

Corresponding to any $f \in F$ we define the maximum modulus $M(r, f)$ on R_+^n by $M(r, f) = \max_{|z_i| = r_i} |f(z)|$. Throughout this section $M(r, k)$ and $M^*(r, k)$ will respectively denote the maximum modulus of the functions $(f * g)^{(k)}$ and $f^{(k)} * g^{(k)}$ on $|z_i| = r_i$.

Evidently $M(r, \bar{O}) = M^*(r, \bar{O})$ for any $r \in R_+^n$. We also see that

$$(2.5) \qquad M(r, k) \leqslant k!\,\frac{M(R, \bar{O})}{(R-r)^k} \quad \text{for } \bar{O} \leqslant r << R \text{ and } k \in I^n.$$

Theorem 1. For $f, g \in F$, as defined by (2.1) and (2.3)

$$(2.6) \qquad M^*(r, k) \leqslant \frac{k!\,R^k M(r, k)}{(R-r)^k} \leqslant \frac{(k!)^2 R^k M(R', \bar{O})}{(R-r)^k (R'-R)^k}$$

where $\bar{O} < r << R << R'$ and $k \in I^n$

Proof. We have

$$\frac{\partial^{\|k\|}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \left\{ z^k (f(z) * g(z))^k \right\} = f^{(k)} * g^{(k)}$$

Then, for any z such that $|z_i| = r_i$ $i = 1 \ldots n$ by Cauchy's Integral Formula

$$f^{(k)}(z) * g^{(k)}(z) = \frac{k!}{(2\pi i)^n} \int_C \frac{t^k (f(t) * g(t))^{(k)}}{(t-z)^{k+1}} dt_1 \ldots dt_n$$

where $C = C_1 \times \ldots \times C_n$, $C_i = |t_i - z_i| = R_i - r_i$, $i = 1. \ldots n$.

Therefore

$$M^*(r, k) \leqslant \frac{k!\,R^k M(r, k)}{(R-r)^k}$$

The other part immediately follows from (2 5)

3. Now corresponding to $f \in F$, we define the function maximum $\mu_f(r)$ and central indices $\nu_j(r)$ of $\mu_f(r)$ $j = 1, \ldots, n$ on R^n by

$$\mu_f(r) = \underset{m \in I^n}{\text{Max}} \{ | a_m | r^m \}$$

$$\nu_i(r) = \text{Max} \{ m_i : | a_m | r^m = \mu_f(r) \quad i = 1, \ldots, n \}$$

We call $\nu = (\nu_1, \ldots, \nu_n)$ as the rank of the maximum term $\mu_f(r)$. We shall throughout denote by $\mu(r, k)$ and $\mu^*(r, k)$ the maximum terms of $((f * g)(z))^{(k)}$ and $f^{(k)}(z) * g^{(k)}(z)$. Then

$$(3.1) \qquad \mu(r, k) = \max_{m \in I^n} \left[\prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i - 1} (m_i - j) \right\} | a_m b_m | r^{m-k} \right]$$

and

$$(3.2) \qquad \mu^*(r, k) = \max_{m \in I^n} \left[\prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i - 1} (m_i - j)^2 \right\} | a_m b_m | r^{m-k} \right]$$

Also let

$$\nu_j = \nu_j(r, k) \text{ and } \nu_j^* = \nu_j^*(r, k) \quad i=1, \ldots, n$$

be the central indices of $\mu(r, k)$ and $\mu^*(r, k)$ respectively and

$\nu = \nu(r, k) = (\nu_1, \ldots, \nu_n)$ and $\nu_1^* = \nu_n^*(r, k) = (\nu_1^*, \ldots, \nu_n^*)$ be their respective ranks.

In this section we obtain a few relations between $\mu^*(r, k)$ and $\mu(r,k)$ which give us more information about the class of entire functions defined by (2.1). For any $f \in F$, let D_1 be the set of discontinuities of ν in $| C^n |$ where $\nu = (\nu_1 \cdots, \nu_n)$ is the rank of $\mu_f(r)$. Also let S denote the set of all $r \in | C^n |$ at which $\mu_f(r)$ is attained by more than one term of the series

$$(3.3) \qquad \sum_{\| m \| = 0}^{\infty} a_m r^m$$

[3, J. Gopal Krishna] had shown that D_1 and S are identical.
Hence for $r \in | C^n | - D_1$, $\mu_f(r)$ is attained by only one term of the series (3.3) and the position of that term is $\nu = (\nu_1, \ldots, \nu_n)$.

Theoram 2. For $r \in | C^n | - D_1 \cup D_1^*$ and $k \in I^n$

$$(3.4)\qquad \prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-1}(\nu_i-j)\right\} \leqslant \frac{\mu^*(r,k)}{\mu(r,k)} \leqslant \prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-j}(\nu_{i*}-j)\right\}$$

where D_1 and D_1^* denote the set of discontinuities of ν and $\nu*$ in $|C_{I^*}^n|$, ν and $\nu*$ are he ranks of $\mu(r,k)$ and $\mu^*(r,k)$

Proof. From (3.1)

$$(3.5)\qquad \mu(r,k) \geqslant \frac{\mu^*(r,k)}{\prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-1}(\nu_{i*}-j)\right\}}$$

Also from (3.2)

$$(3.6)\qquad \mu^*(r,k) \geqslant \prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-1}(\nu_i-j)^2\right\} | a_\nu b_\nu | r^{\nu-k}$$

$$=\mu(r,k) \prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-1}(\nu_i-j)\right\}.$$

Hence the the theorem follows from (3.5) and (3.6).

Theorem 3. If ν (r,k) and ν $(r,k+\bar{1})$ be the ranks of $\mu(r,k)$ and $\mu(r,k+\bar{1})$. Then

$$(3.7)\qquad \prod_{i=1}^{n}\left\{\nu_i(r,k)-k_i\right\} \leqslant \frac{\mu(r,k+\bar{1})}{\mu(r,k)}\ \pi r$$

$$\leqslant \prod_{i=1}^{n}\left\{\nu_i(r,k+\bar{1})-k_i\right\}$$

for $r \in |C^n| - D_1 \cup D_2$, $k \in I^n$, where D_1 and D_2 are the set of discontinuities of ν (r,k) and $\nu(r,k+\bar{1})$ in $|C^n|$.

Proof. From (3.1), we have

$$\mu(r,k+\bar{1}) = \prod_{i=1}^{n}\left[\nu_i(r,k+\bar{1}) \dots, \left\{\nu_i(r,k+\bar{1})-k_i\right\}\right]$$

$$| a_{\nu(r,k+\bar{1})} b_{\nu(r,k+\bar{1})} | r^{\nu(r,k+\bar{1})-k-\bar{1}}$$

Therefore,

$$\mu(r,k+\bar{1}) \leqslant \mu(r,k) \frac{\prod_{i=1}^{n} \left\{ \nu_i (r,k+\bar{1}) - k_i \right\}}{\prod_{i=1}^{n} r_i}$$

i. e.

(3.8) $$\frac{\mu(r,k+\bar{1})}{\mu(r,k)} \pi r \leqslant \prod_{i=1}^{n} \left\{ \nu_i (r,k+\bar{1}) - k_i \right\}$$

Again

$$\mu(r,k+\bar{1}) \geqslant \prod_{i=1}^{n} [\ \nu_i\ r,k) \cdots \left\{ \nu_i (r,k) - k_i \right\}]$$

$$| a_{\nu}(r,k) b_{\nu} (r,k) |\ r^{\nu (r,k) - k - \bar{1}}$$

(3.9) $$= \prod_{i=1}^{n} \left\{ \nu(r,k) - k_i \right\} \frac{\mu(r,k)}{\pi r}$$

(3.8) together with (3.9) proves (3.7)

Corollaries. 1) $\prod_{i=1}^{n} \nu_i (r,\bar{O}) \leqslant \prod_{i=1}^{n} \nu_i (r,\bar{1}) \leqslant \cdots$

2) $\dfrac{\mu(r,\bar{1})}{\mu(r,\bar{O})} \leqslant \dfrac{\mu(r,2)}{\mu(r,\bar{1})} \leqslant \cdots$

3) Putting $k = \bar{O}, \ldots \overline{p\text{-}1}$ successively in (3.7) and from above we get

$$\prod_{i=1}^{n} \nu_i (r,\bar{O}) \leqslant \left\{ \frac{\mu(r,\bar{p})}{\mu(r,\bar{O})} \right\}^{1/p} \leqslant \prod_{i=1}^{n} \nu_i (r,\bar{p})$$

If we do not delete the set of discontinuities of ν and ν^*, the above theorems take the following forms whose proofs follows in the same line as those of the above.

Theoram 2′ For any $r \in | C^n |$, $k \in I^n$

$$\prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i-1} (p_i - j) \right. \leqslant \frac{\mu^*(r,k)}{\mu(r,k)} \leqslant \prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i\ 1} (\nu_i{*} - j) \right\}$$

where $p=(p_1, \ ,p_n)$ is a position of occurrence of $\mu(r,k)$ and $\nu^*(r,k)=(\nu_1{*},\cdots,\nu_n{*})$ is the rank of $\mu^*(r,k)$.

Theorem 3′ If $q=(q_1, \cdots,q_n)$ is a position of occurrence of $\mu(r,k)$ and $\nu=(\nu_1,\cdots\nu_n)$ is the rank of $\mu(r,k+\bar{1})$, then, for any $r \in | C^n |$

$$\prod_{i=1}^{n} \left\{ q_i(r,k) - k_i \right\} \leqslant \frac{\mu(r,k+\bar{1})}{\mu(r,k)} \pi r$$

$$\leqslant \prod_{i=1}^{n} \left\{ \nu_i(r,k+\bar{1}) - k_i \right\}$$

4. **In** this section we shall consider Gol'dberg order and Gol'dberg type of an entire function in C^n. Let $D \subset C^n$ be an arbitrary bounded complete n-circular domain with centre at the origin of coordinates. Then for the entire function f, we define

$M_{f,\ D}(r) = \sup\limits_{z \in D_r} | f(z) |$, $r(>O) \in R$, where the point $z \in D_r$ iff the point $\left(\frac{z_1}{r},\ldots\ldots,\frac{z_n}{r}\right) \in D$. The Gol'dberg order ρ_D and Gol'dberg type σ_D (briefly G-order and G-type) of f w. r. t the domain D are defined by the formulas [1, Fuks P. 339]

$$P_D = \limsup_{r \to \infty} \frac{\log \log M_{f,\ D}(r)}{\log r} \qquad \sigma_D = \limsup_{r \to \infty} \frac{\log M_{f,\ D}(r)}{r^D}$$

It turns out that the G-order ρ_D does not depend on the choice of the domain D while the G-type σ_D does [1, P.339]. It is also known [1, P.339] that for the entire function f

$$(4.1) \qquad P_D = P = \limsup_{\| m \| \to \infty} \frac{\| m \| \log \| m \|}{-\log | a_m |} \quad \text{and}$$

(4.2) $$(e\rho\sigma_D)^{1/\rho} = \limsup_{\|m\|\to\infty} \left\{ \|m\|^{1/\rho} \left(|a_m| \, d_m(D) \right)^{\frac{1}{\|m\|}} \right\}$$

where $$d_m(D) = \sup_{z \in D} \left\{ |z_1|^{m_1} \ldots |z_n|^{m_n} \right\}.$$

Theorem 4 If f is an entire function with G– order ρ ($O < \rho < \infty$), then $f^{(k)}$ $(k \in I^n)$ is also of G-order ρ. Moreover, for a bounded complete n-circular domain D, the entire function f and $f^{(k)}$ have he same G-typc σ_D.

Proof. Let $f(z) = \sum_{\|m\|=o}^{\infty} a_m z^m$ be of G-order ($O < \rho < \infty$). Then, from (2.4)

$$f^{(k)}(z) = \sum_{\|m\|=o}^{\infty} \left[\prod_{i=1}^{n} \left\{ \prod_{j=o}^{k_i-1} (m_i - j) \right\} \right] a_m z^{m-k}$$

Now $$\liminf_{\|m\|\to\infty} \frac{-\log \left[|a_m| \prod_{i=1}^{n} \left\{ \prod_{j=o}^{k_i-1} (m_i - j) \right\} \right]}{\|m\| \log \|m\|}$$

$$= \liminf_{\|m\|\to\infty} \left[\frac{-\log |a_m|}{\|m\| \log \|m\|} - \frac{\log \prod_{i=1}^{n} \left\{ \prod_{j=o}^{k_i-1} (m_i - j) \right\}}{m \log m} \right]$$

$$= \liminf_{\|m\|\to\infty} \frac{-\log |a_m|}{\|m\| \log \|m\|} = \frac{1}{\rho} \quad \text{by (4.1)}$$

Again $$\limsup_{\|m\|\to\infty} \left[\|m\|^{1/\rho} \left\{ |a_m| \, d_m(D) \prod_{i=1}^{n} \left(\prod_{j=o}^{k_i-1} (m_i - j) \right) \right\}^{\frac{1}{\|m\|}} \right]$$

$$= \limsup_{\|m\|\to\infty} \left[\|m\|^{1/\rho} \left[|a_m| \, d_m(D) \right]^{\frac{1}{\|m\|}} \right]$$

which proves the Theorem,

Remark. In a similar way we can prove the following :

For f, g ϵ F, if f * g is of G-order ρ $(O < \rho < \infty)$ and G-type σ_D corresponping to a bounded complete n-circular domain D then $f^{(k)} * g^{(k)}$ is also of G-order ρ and G-type σ_D.

Theroem 5. Let f, g ϵ F be of G-osders ρ_1 and ρ_2 respectively, then f * g ϵ F and satisfy $1/\rho \geqslant 1/\rho_1 + 1/\rho_2$ where ρ is the G-order of f * g.

Proof. That f * g ϵ F is evident. Now,

$$1/\rho = \liminf_{\|m\| \to \infty} \frac{-\log | a_m b_m -}{\|m\| \log \|m\|}$$

$$(4.3) \qquad \geqslant 1/\rho_1 + 1/\rho_2 \qquad \text{by } (4\,1)$$

Corollary. If f, g ϵ F be of G-orders ρ_1 and ρ_2 respectively then $f^{(k)} * g^{(k)}$ ϵ F and be of G-order ρ satisfying (4.3)

5. Let f be an entire function and M (r) be its maximum modulus. Let B_f denote the set (may be empty) of all points $\alpha \in R^n_+$ such that

$$\log M(r) < r_1^{\alpha_1} \ldots + r_n^{\alpha_n} \text{ for } \|r\| \to \infty .$$

The boundary ∂B_f of the set B_f is called the order of f a d any point $\rho \in \partial B_f$ is called an order point. We say that f is of finite or infinite order according as B_f is non empty or empty. Evidently, for any $\rho \in \partial B_f$, $\rho \geqslant \bar{O}$. An order point ρ is said to be positive if $\rho >> \bar{O}$.

Next, let $\rho \in \partial B_f$ $(\rho >> \bar{O})$ and $T_f = T_f(\rho)$ denote the set (may be emyty) of all points $\beta \in R^n_+$ such that $\log M(r) < \beta_1 r_1^{\rho_1} + \ldots + \beta_n r_n^{\rho_n}$ for $\|r\| \to \infty$.

The boundary ∂T_f of the set T_f is called the type of f corresponding to the order point ρ. A point $\sigma \in \partial T_f$ is called a type point of f. A type point σ is called positive if $\sigma >> \bar{O}$.

We say that f is of finite or infinite type according as T_f is non-empty or empty.

Theorem 6. The entire functions f and $f^{(k)}$ ($k \in I^n$) have the same set of positive order points. Moreover for a positive order point they have the same set of positive type points.

Proof. Let f be as in (2.1) and let ρ be an order point of f. It is known [P. 137 Ronkin] that ρ ($>> \bar{O}$) is an order point of f iff

$$(5.1) \qquad \limsup_{\|m\| \to \infty} \frac{\frac{m_1}{\rho_1}\log m_1 + \dots + \frac{m_n}{\rho_n}\log m_n}{-\log |a_m|} = 1$$

Now,

$$\limsup_{\|m\| \to \infty} \frac{\frac{m_1}{\rho_1}\log m_1 + \dots + \frac{m_n}{\rho_n}\log m_n}{-\log \left[\prod_{i=1}^{n} \left(\prod_{j=o}^{k_i-1} (m_i - j) \right) |a_m| \right]}$$

$$= 1 \Big/ \liminf_{\|m\| \to \infty} \frac{-\log \left[\prod_{i=1}^{n} \left(\prod_{j=o}^{k_i-1} (m_i - j) \right) |a_m| \right]}{\frac{m_1}{\rho_1}\log m_1 + \dots + \frac{m_n}{\rho_n}\log m_n}$$

$$= 1 \Big/ \liminf_{\|m\| \to \infty} \frac{-\log |a_m|}{\frac{m_1}{\rho_1}\log m_1 + \dots + \frac{m_n}{\rho_n}\log m_n} = 1 \qquad \text{by } (5.1)$$

By reversing the step we get the converse part which settles the first part.

It is known [P.139, Ronkin] that $\sigma(>>\bar{O})$ is a type point of f for a positive order point ρ iff

$$(5.2) \qquad \limsup_{\|m\| \to \infty} \left\{ |a_m| \prod_{i=1}^{n} \left(\frac{m_i}{e\sigma_i} \right)^{m_i/\rho_i} \right\}^{\frac{1}{\|m\|}} = 1$$

Let $\sigma(>>\bar{O})$ be a type point of f the positive order point ρ so (5.2) holds.

Now,

$$\limsup_{\|m\|\to\infty} \left[|a_m| \prod_{i=1}^{n} \left(\frac{m_i}{e\sigma_i \rho_i}\right)^{m_i/\rho_i} \prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i-1} (m_i - j) \right\} \right]^{\frac{1}{\|m\|}}$$

(5.2) $$= \limsup_{\|m\|\to\infty} \left[|a_m| \prod_{i=1}^{n} \left(\frac{m_i}{e\sigma_i \rho_i}\right)^{m_i/\rho'_i} \right]^{\frac{1}{\|m\|}} = 1 \text{ by}$$

Hence σ is a type point of $f^{(k)}$ for the order point ρ. Reversing we get converse part. This completes the proof.

Theorem 7. If f, g ϵ F then f $*$ g and $f^{(k)} * g^{(k)}$ ($k \epsilon I^n$) have the same set of positive order points. Further, for a positive order point they have the same set of positive type point.

Proof. The proof is exactly similar to that of Theorem 6.

Remark. We observe that while our order points and type points are subsets in R^n_+ the G-order and G-type of an entire function are simply non-negative real numbers.

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