Jour. Pure Math
Vol. 6 (1987-88), PP15—19

ON REGULAR RINGS

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1. Regular rings were first introduced by J. V. Neumann and were subsequently studied by many prominent mathematicians. In this paper we have studied some properties of regular rings. Among other things we have proved that a regular ring is primitive iff it is prime. Further we have studied the nature of subdirectly irreducible regular rings

2. Here rings are all associative. Unless otherwise stated by an ideal we mean a both sided ideal. Let R be a ring and M be a right R-module. we shall say M is faithful if $Mr = (O)$, $r \in R$ implies $r = O$.

Let R be a ring and M be a right R-module. If $MR \neq (O)$ and if the only submodules of M are (O) and M, then M is called irreducible.

A ring will be called right primitive, if it has a faithful irreducible right R-module.

A ring R will be called prime, if for any to both sided ideals I and J, $IJ = (O)$ implies either $I = (O)$ or $J = (O)$.

A ring R (not necessarily with 1) will be called regular if for every a in R, $axa = a$ is solvable in R.

A ring R is said to be subdirectly irreducible if the intersection of all its non-null both sided ideals is different from the zero ideal.

An element a in a ring R is said to be right quasi-regular if there is an $a' \in R$ such that $a+a'+aa' = O$.

We say that a right ideal of R is right quasiregular if each of its elements is right quasiregular.

The radical of R written as J (R) is the set of all elements of R which annihilate all the irreducible R-modules.

If $J(R) = (O)$ then R is called semisimple. It can be proved $J(R)$ is a right quasiregular ideal.

Proposition 1. Any regular ring R is semisimple.

Proof : Let R be a regular ring and $a \in J(R)$. Now R being regular, $\exists\, x \in R$ such that $axa = a$. Put $ax = e$. Thus $e^2 = e$. Also $e \in J(R) \Rightarrow -e \in J(R)$. As $J(R)$ is a right quasiregular ideal, there exist $f \in J(R)$, such that $-e+f-ef = O \Rightarrow e = f - ef \Rightarrow e = e^2 = ef - ef = O$. Then $a = ea = O$. Thus R is semisimple.

Theorem 1. Any subdirectly irreducible regular ring is primitive.

Proof : Let R be a subdirectly irreducible regular ring. As R is regular, it is semisimple. Then $\bigcap A(M) = (O)$, where M runs over all irreducible R-modules and A(M) denotes its annihilator. It can be easily verified A (M) is a both sided ideal.

As R is subdirectly irreducible, the intersection of all its nonnull ideal is not a zero ideal. Hence there exists $A(M_1)$ in R with $A(M_1) = (O)$ where M_1 is an irreducible R-module.

Thus R is primitive.

Corollary : Any subdircetly irreducible regular ring is prime.

Proof : It follows immediately from the theorem proved above, as every primitive rlng is prime.

Converse is also true, and we give it in the following theorem.

Theorem 2. Any prime regular ring is subdirectly irreducible.

Proof : Let R be a prime regular ring. Let X denote the set of all non-null ideals in R

Consider the power set $P(X)$. $P(X)$ is a poset w.r.t usual set inclusion relation.

We define a property P on $P(X)$ as follows :

An element $A \in P(X)$ is said to have property P iff

$$\bigcap \{ I_\alpha, I_\alpha \in A \} \neq (O)$$

Evidently the minimal elements of $P(X)$ have the property P as they are of the form $\{I\}$, where I is a non-null ideal of R. Suppose the property P holds for every element, $Z < Y$ and

$Z \neq Y$ in $P(\mathcal{X})$. We shall prove that Y also has the property P. Let Z' denote the complement of Z in Y. Then $Z' < Y$ and $Z' \neq Y$ in $P(\mathcal{X})$.

By induction hypothesis $\cap \{ I_\alpha, I_\alpha \in Z \} \neq (O)$.

Denote it by S_1.

And $\cap \{ P_\alpha, P_\alpha \in Z' \} \neq (O)$ Denote it by S_2.

Then $\cap \{ T_\alpha, T_\alpha \in Y \} = S_1 \cap S_2$ (1)

Now $S_1 \neq (O)$, $S_2 \neq (O)$ implies $S_1 \cap S_2 \neq (O)$

If possible, let $S_1 \cap S_2 = (O)$.

Let $p \in S_1 \cap S_2$, then by regularity $p x p = p$ for some x in R. Now S_2 being both sided ideal $x p \in S_2 \Rightarrow p = p\ (x p) \in S_1S_2$, that is $S_1 \cap S_2 \subseteq S_1S_2$

Also $S_1S_2 \subseteq S_1 \cap S_2$. Thus $S_1 \cap S_2 = S_1S_2$.

Now R being prime, $S_1S_2 = (O) \Rightarrow S_1 = (O)$ or $S_2 = (O)$ contradiction shows that $S_1 \cap S_2 \neq (O)$.

Then from (1) we get Y has the property P. By induction every element of $P(\mathcal{X})$ has property P.

As $\chi \in P(\chi)$, χ has the property P.

The intersection of all non-null ideals of R is not a zero ideal. Therefore R is subdirectly irreducible.

Combining Theorem 1 and Theorem 2 we have

Corollary : Any prime regular ring is primitive.

This answers a problem posed by I. Kaplansky [7].

3. A ring in which every both sided principal ideal is generated by a central idempotent, is called a biregular ring.

Theorem 3. Any simple ring with 1, is a subdirectly irreducible biregular rlng.

Proof : The only both sided non-null ideal is the ring R. So R is subdirectly irreducible. Evidently R is biregular.

Theorem 4 : Any subdirectly irreducible biregular ring is a simple ring with 1.

Proof : Let R be a subdirectly irreducible biregular ring. If possible let R be not simple.

Let $P = \cap \{ I_\alpha : I_\alpha \neq (O) \}$. Then $P \neq (O)$, as R is subdirectly irreducible.

Now $P \neq R$, for otherwise $P = R$ implies R is simple.

Thus P is a minimal ideal in R.

As $P \neq (O)$, there exists an element $a \in P / a \neq O \Rightarrow (a) \subseteq P$. As P is minimal $P = (a)$. As R is biregular, $P = (e)$, where e is a central idempotent. Let $(e)^*$ denote the annihilator of (e). Now $e \notin (e)^* \Rightarrow (e)^* \neq R$. If B is an ideal properly containing $(e)^*$, there exists an idempotent $f \neq O$ in the centre of R with $f \in B$, $f \notin (e)^*$.

But $(e) \subseteq (f)$. So $(e) \subseteq B$. Thus $(e) + (e)^* \subseteq B$ and so by Pierce decomposition $B = R$ and $(e)^*$ is maximal and hence $(e)^* \neq (O)$.

Now P being intersection of all non-null ideals $(e) \subseteq (e)^* \Rightarrow e = O$. We arrive at a contradiction. Thus R is simple.

Now as the centre is not a zero ring it is a field [2], and hence the centre contains an identity 1.

Thus $1 \in$ centre $R \subseteq R$.

We shall now prove that 1 is the identity of the whole ring R.

Let $a \in R$. Then $(a) = (e)$ for some central idempotent e. Thus $ae = a$.

$\therefore a.1 = ae.1 = ae = a$

$\therefore$ 1 is the identity of R.

Thus R is a simple ring with 1.

4. A ring R is said to be strongly regular if for each element $a \in R$, there exists an $x \in R$ with $a^2x = a$.

It can be proved that strong regularity implies regularity and biregularity.

Proposition 5. Any skew field is a subdirectly irreducible strongly regular ring.

Proof is obvious.

Converse is also true.

Proposition 6. Any subdirectly irreducible strongly regular ring is a skew field.

Proof: Let R be any subdirectly irreducible strongly regular ring.

Then by Theorem 4, R is a simple ring with 1. Now as R is strongly regular, every idempotent is central, and as R is simple its centre is a field. Thus R contains only two idempotents viz O and 1. Let $O \neq a \in R$. Then $\exists\, x \in R \mid axa = a \Rightarrow ax$ is an idempotent $a \neq O$ $\Rightarrow ax \neq O$ thus $ax = 1$. Thus R is a skew field.

In conclusion, I thank Dr. A. K. Dasgupta and Dr. S. Chandra for their co-operation in the preparation of the paper. My thanks are also due to Prof. S P. Bandyopadhyay for his suggestions and comments.

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Received
24.8.1987

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