Jour Pure Math
Vol 6 (1987—88) PP 9—13

COMPATIBLE TOPOLOGIES OF A GROUP AND THOSE OF ITS LATTICE OF SUBGROUPS—(II).

A. DAS GUPTA

1. It has been found [3] that the set of all compatible topologies of a group G and those of its lattioe L (G) of subgroups form complete lattices T and L (T) respectiaely. Also, it has been proved that for a topology $t \in T$, we have a topojogy $t^* \in L(T$ and for the topology $t^* \in L(T)$ we have a topology $t' \in T$. Also it has been shown that $t' \leqslant t$. In this paper we have studied the condition under which t and t' are equal. We note that if $\cap U' = e, \forall U' \in \Sigma'$ where Σ' is the complete system of neighbourhoods of the identity e of the group G for tha topology t', then G with the topology t' is a Hausdorff space.

2. Let $t \in T$ and t' be that element of T which corresponds to $t^* \in L(T)$, where t^* is the corresponding element of t. [3]

Then we have the following :

Theorem—1 Let identity be the only common element of the complete system of neighbourhoods of identity for the topology t'. If t is compact, then $t = t'$ holds.

Proof : Consider the function $f : (G, t) \to (G, t')$ defined by $f(x) = x, \ \forall x \in G$ i. e. the identity function on G.

Now, the identity mapping f is algebrically an isomorphic mapping. Furthermore, the mapping f is continuous, since $t' \leqslant t$.

Henee, as t is compact and as $\cap U' = e, \ \forall U' \in \Sigma'$, where Σ' is the complete system of neighbourhoods of identity for the topology t', it follows that f is a homeomorphism.

Therefore, $t = t'$.

The converse of this theorem is not true i. e. $t' = t$ and $\cap U' = e, \forall U' \in \Sigma'$, does not imply that t is compact.

This can be shown from the following example.

Evample : Let G be the additive group of integers. Let p be a prime number, we denote by U_k the set of all integers, which are divisible by p^k.

We take for a complete system of neighbourhoods of zero, the totality Σ of all sets U_k, $k = 1, 2, 3, \ldots$. It can be easily verified that Σ becomes a complete system of neighbourhoods of identity for which G is a topological group.

Let us denote this topology by t.

Now, each U_k is algebrically a subgroup of G.

For, if $a, b \in U_k$ then $a-b \in U_k$.

From the topology t of T, we get the topology t^* in L (T) and from t^* we get t' in T.

Here obviously $t = t'$.

The topology t is not compact.

If posible, let t be compact.

Then $\{ U_k + n \}$ for $n = \pm 1, \pm 2, \pm 3, \ldots$ is an open cover of G.

From this open cover we can find a finite subcover of G.

That is, there exits integers, $n_1, n_2, \ldots n_m$ such that $G \subset \{U_k + n_1\} \cup \{U_k + n_2\} \cup \ldots \cup \{U_k + n_m\}$.

Let N be an integer such that $N - n_i$, $i = 1, 2 \ldots m$ is not divisible by p^k.

Let $N \in \{U_k + n_j\}$, where $1 \leqslant j \leqslant m$.

Then $a + n_j = N$ for some $a \in U_k$, or, $a = N - n_j \epsilon U_k$. Contradiction as $N - n_j$ is not divisible by p^k and so, $a \notin U_k$.

3. We define a generalised topological group.

Definition :

A set G of elements is called a generalised topological group if

(1) G is an abstract group

(2) G is a topological space.

(3) The group operations in G are continuous in the topological space G.

Let G be a group and t be not the weakest compatible topology of G, so that G with the topology t is a generalised topological group.

Let t be a compact topology.

Then we can show that there exists a topology $\bar{t}$, for which the complete system of neighbourhoods of identity is $\bar{\Sigma} = \{ N, G \}$, where N is a normal subgroup and $\bar{t} \leqslant t$.

For, from t we get $t' \mid t' \leqslant t$ [3].

Let H be any neighbourhood of identity belonging to t′.

Let $N = \bigcap_{x \in G} xHx^{-1}$. Here, $N \neq G$.

Since t is compact, there exists a neighbourhood V of $t \mid xVx^{-1} \subset H$ for all $x \in G$.

Hence, $N \subset x^{-1} Hx$, $\forall\, x \in G$. So, $V \subset N$.

Hence, N is an open normal subgroup of G.

We consider the set $\bar{\Sigma} = \{N, G\}$.

Then it can be shown that $\bar{\Sigma}$ satisfies all the five conditions :—

(1) Identity belongs to all the sets.

(2) The intersection of any two sets of the system $\bar{\Sigma}$ contains a third set of the system $\bar{\Sigma}$.

(3) For every set U of the system $\bar{\Sigma}$, there exists a set V of the same system, such that $VV^{-1} \subset U$.

(4) For every set U of the system $\bar{\Sigma}$ and an element $a \in U$, there exists a set V of the system $\bar{\Sigma}$, such that $Va \subset U$.

(5) If U is a set of the system $\overline{\Sigma}$ introduced above and 'a' an arbitrary element of the group G, then there exists a set V of the system $\overline{\Sigma}$, such that $a^{-1}Va \subset U$.

Hence $\overline{\Sigma}$ generates a compatible topology which we denote by $\bar{t}$. Also, it can be easily shown that $\bar{t} \leqslant t' \leqslant t$, where t and t′ have been introduced before.

Now, if G is a simple group, then the compact topology t is an atom in the lattice T.

For, let t_1 be any compatible topology such that $t_1 \leqslant t$.

Let Σ_1 be the complete system of neighbourhoods of identity of the topology t_1.

Then $\bigcap U = e.\ \forall\, U \in \Sigma_1$, since G is simple.

Hence G, with the topology t_1 is a Hausdorff space.

Also, t is compact. Hence by theorem 1, $t_1 = t$ holds.

Therefore, t must be an atom.

Hence, we have $\bar{t} = t' = t$ holds, if G is simple.

Also, $\bar{t}$ is a discrete topology as $N=(e)$, the identity subgroup.

Hence, G with the topology t, is a compact discrete space. So, G is a finite group and T must be a two element lattice, as t is discrete and is an atom.

Therefore, we have the following theorem :

Theorem—2 :

If G be a simple group and $t \in T$ be a compact topology, then T is a two element lattice and G is finite.

Acknowledgement :

The author wishes to thank Professor S. P. Bandyopadhyaya, C. U. for suggesting the problem.

REFERENCES

[1] Birkhoff, G. Lattice Theory Amer. Math. Soc. Coll. Publ. 1963

[2] Chatterjee, B C. Abstract Algeqra
Das Gupta & Co. (Pvt) Ltd.

[3] Das Gupta, A. On compatible topologies of a group and those of lattice of subgroups.
(Communicated .

[4] Hussian, T. Introduction to Topological Groups.
W. B. Saunders Company.
Philadelphia and London 1966.

Department of Mathematics
B. E. College,
Shibpur, Howrah.

Received
12.5.1987