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π-GROUP CONGRUENCE ON AN EVENTUALLY REGULAR SEMIGROUP

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1. INTRODUCTION

A semigroup is eventually regular if for every $a \in S$ there exists a positive integer m such $a^m \in a^m Sa^m$. We shall denote by Reg*S* the set of all regular elements of S and by E(S) the set of all idempotents of S. If $x \in$ RegS, V(x) will denote the set of inverses of x.

An eventually regular semigroup S is called a π-group if $|E(S)| = 1$. It is easy to see that a π-group is nil-extension of a group. *A* nonempty subset *A* of a semigroup S is said to be *N*-subset if for any $x,y \in S^1$, $xAy \subset A$ whenever $xAy \cap A \neq \emptyset$. If a N-subset *A* of S is a subsemigroup of S, we will say that *A* is a *N*-subsemigroup of S. If $E(S) \subset A$, we say *A* is full.

Congruences on an eventually regular semigroup are discussed in [4], [5], [6] using the similar method which is used in regular semigroup. In this paper we first give some properties of the *N*-subset of S and then we describe the π–group congruence on an eventually regular semigroup. At the end we will prove that the π–group congruence is just the group congruence when S is regular and this will give a new description of the group congruence on a regular semigroup.

2. PROPERTIES OF *N-SUBSETS*

Theorem 2.1. Let *S* be a semigroup.
(1) If ρ is a congruence on *S*, then every ρ-class of *S* is a *N*-subset of *S*.
(2) Let *A* be a *N*-subset of *S*, define a binary relation on *S* by

$$Q_A = \{(x,y) \in S \times S : x = y \text{ or } x,y \in A\}$$

Let ρ_A denote the congruence generated by Q_A, then *A* is a congruence class of ρ_A and ρ_A is the smallest congruence on *S* which makes *A* as its congruence class.

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(3) If $\{A_i : i \in I\}$ is a family of N-subsets of S and $A = \bigcap_{i \in I} A_i$ is nonempty, then A is a *N-subset of S.*

(4) Let S_1 be a semigroup and $\phi : S \rightarrow S_1$ be an epimorphism from S to S_1. Then the inverse image $A_1\phi^{-1}$ of a N-subset A_1 of S_1 is a N-subset of S.

(5) Let A be a N-subset of S. Then A is a N-subsemigroup of S iff there exist x and y in A *such that* $xy \in A$.

(6) If the intersection of a subsemigroup T and a N-subset A of S is nonempty, then $A \cap T$ is a N-subset of T.

Proof. (1) For every $a \in S$, $a\rho$ denotes the ρ-class containing a. Let $x,y \in S^1$ such that $x(a\rho)y \cap (a\rho) \neq \emptyset$, then $a\rho b$ for some $b \in a\rho$ and $(xby)\rho a$. Thus for every $c \in a\rho = b\rho$, $(xcy)\rho\ (xby)\rho a$ and $x(a\rho)y \subset a\rho$

(2) First it is easy to see $a\rho_A b$ for $a,b \in A$. Suppose now $x \in S$ and $x\rho a$ for some $a \in A$. Then $x = a$ or there exist $z_1, z_2, \ldots, z_{n-1} \in S$ such that $(z_i, z_{i+1}) \in Q_A^C$ for $i = 0,1,2,\ldots, n-1$, *where* $z_0 = a$, $z_n = x$ and $z_i \neq z_{i+1}$ for $i = 0,1,\ldots n-1$. If $x = a$ then $x \in A$.

Otherwise, if $x \neq a$, since $(a = z_l) \in Q_A^C$, there exist $x_l, y_l \in S_1$ such that $a = x_l b_l y_l$, $z_l = x_l c_l y_l$, where $(b_l c_l) \in Q_A$ for some $b_l, c_l \in S$. Thus $b_l, c_l \in A$ since $a \neq z_l$ implies $b_l \neq c_l$. Then

$$a = x_l b_l y_l \in x_l A y_l \cap A \neq \emptyset \text{ and thus } x_l A y_l \subset A$$

So that $z_l = z_l c_l y_l \in A$.

Using the same deduction we obtain that $z_2, z_3, \ldots z_{n-1} \in A$, so that $x \in A$ and then A is a congruence class of ρA.

Now suppose that ρ is a congruence on S such that A is a congruence class of ρ. Suppose $x,y \in S$ and $(x,y) \in Q_A$, then $x = y$ or $x,y \in A$. So that $(x,y) \in \rho$, thus $Q_A \subset \rho$ and then $\rho A \subset \rho$.

(3) Follows from the definition of N-subset easily.

(4) Since ϕ is an epimorphism, the binary relation defined by $\rho = \{(x,y) : (x\phi, y\phi) \in \rho A_1\}$ is a congruence on S. Let $a,b \in A_1\phi^{-1}$, then $(a\phi, b\phi), \in \rho A_1$, and thus $(a,b) \in \rho$. On the other hand, if $a \in A_1\phi^{-1}$, $x \in S$ and $x\rho a$, then $(x\phi, y\phi) \in \rho A_1$, that is $x\phi \in (a\phi)\, \rho A_1 = A_1$, thus $x \in A_1\phi^{-1}$. So that $A_1\phi^{-1}$ is a congruence class of ρ and then $A_1\phi^{-1}$ is a N-subset of S by (1).

(5) Suppose $x,y \in A$ such that $x,y \in A \cap A \quad \emptyset$. So that $x A \subset A$ since A is a N- subset of S. Now for every $b \in A$, $xb \in A$ which implies $Ab \cap A \neq \emptyset$ and then $Ab \subset A$. This A is a subsemigroup of S.

(6) Suppose $x,y \in T^1$ such that $[x(A \cap T)y] \cap (A \cap T) \quad \emptyset$ and thus $x(A \cap T)y \subset xAy \subset A$, so that $x(A \cap T)y \subset A \cap T$ since obviously $x(A \cap T)y \subset T$.

Corollary 2.2. If a N-subset A of a semigroup S contains idempotents, then it is a N-subsemigroup. Conversely, every N-subsemigroup of an eventually regular semigroup contains idempotents.

Proof. Let e = $e^2 \in A$, then by (5) of Theorem 2.1, A is a N-subsemigroup of S. Conversely, If A is a N-subsemigroup of S, and $a \in A$, then $a^2 \in A$. By (2) of Theorem 2.1, $A = a\rho_A \in E(S/\rho_A)$. So that there exists an idempotent e $\in E(S)$ such that $e\rho_A = a\rho_A = A$ since S is eventually regular.

3. π-GROUP CONGRUENCE ON AN EVENTUALLY REGULAR SEMIGROUP

Definition 3.1. Let S be a semigroup and $a,b \in S$, we say b devides $a(b/a)$, if $a = xby$ for some $x,y \in S^1$.

In this section S will be always an eventually regular semigroup unless otherwise stated.

Lemma 3.1. Let m be the positive integer such that $a^m \in$ Reg S. and If $(a^m)' \in V(a^m)$, then $a^i\,(a^m)'a^j \in E(S)$, $i + j = m$.

Lemma 3.2. Let A be a full N-subsemigroup of S and $a \in S$. If there exists an element $u \in A$ such that u/a then $(a\vartheta)\rho_A = a\rho_A = (\vartheta a)\rho_A$ for every $\vartheta \in A$.

Proof. Since A is a full N-subsemigroup of S, then it is the idempotent of S/ρ_A by Theorem 2.1 (2).

From u/a, we have $a = xuy$ for some, $x,y \in S^1$. Let $n \in N$ such that $y^n \in$ Reg$S$ and $(y^n)' \in V(y^n)$.

$$
\begin{aligned}
(a\vartheta)\rho_A &= (xuy\vartheta)\rho_A \\
&= [xuy\,(y^n)'y^m\vartheta]\rho_A && \text{(since } (y^n)'y^m \in E(S) \subset A\text{)} \\
&= [xuy\,(y^n)'y^m]\rho_A && \text{(since } \vartheta \in A\text{)} \\
&= [xu\,(y\,(y^n)'y^{n-1})y]\rho_A \\
&= (xuy)\rho_A && \text{(since } y\,(y^n)'y^{n-1} \in E(S) \subset A\text{)} \\
&= a\rho_A
\end{aligned}
$$

Similarly, $a\rho_A = (\vartheta a)\,\rho_A$.

Lemma 3.3. Let S be a semigroup and A be a N-subsemigroup of S. Then for any $a,b \in S$, if $a\rho_A b$ and $a \neq b$, then there exist $u,\vartheta \in A$ such that u/a and u/b.

Proof. By the definition of ρ_A, there exist $z_1, z_2, \ldots, z_{n-1} \in S$ such that $(z_i, z_{i+1}) \in Q_A^C$, and $z_i \neq z_{i+1}$ for $i = 0,1,\ldots, n-1$. where $z_0 = a, z_n = b$.

Since $(a,z_1) \in Q_A^C$, there exist $x,y \in S_1$, $(u,w) \in Q_A$ such that $a_1 = xuy$, $z_1 = xwy$. Thus $u \neq w$ since $a \neq z_1$ and then $u,w \in A$ and u/a.

Using the same method we can prove that there exists $\vartheta \in A$ such that ϑ/b.

Theorem 3.4. Let A be a full N-subsemigroup of S. Then

(1) ρ_A.ia a π-group congruence on S.

(2) For any $a,b \in S$ $a\rho_A b$ iff $a = b$ or there exist $u,\vartheta,\omega \in A$ and e $\in E(S)$ such that $u\,|\,a$, $\vartheta\,|\,b$ and be $= wa$.

Proof. (1) Since A is a ρ_A-class by Theorem 2.1 and S/ρ_A is obvious eventually regular. Then ρ_A is a π —group congruence of S since $E(S) \subset A$.

(2) If $a\rho_A b$ and $a \neq b$, by Lemma 3.3, there exist $u, \vartheta \in A$ such that $u|a$, $\vartheta|b$ Now Let m be the positive integer such that $a^m \in$ Reg$S$ and $(a^m)' \in V(a^m)$, then

$[a^m (a^m)']\ \rho_A\ [ba^{m-1} (a^m)']$ and thus $ba^{m-1} (a^m)' \in A$

Since $a^m (a^m)' \subset E(S) \subset A$ and A is a ρ_A-class. Let $e = a^{m-1} (a^m)' a \in E(S)$, $w = ba^{m-1} (a^m)'$, then $wa = be$.

Conversly, suppose a = b the $a\rho_A b$. Otherwise, there exist u, v $w \in$ A and e $\in$ E (S), such that $u|a$, $\vartheta|b$, $wa = be$. By lemma 3.2, $a\rho_A = (wa)\ \rho_A = (be)\ \rho_A = b\rho_A$.

Theorem 3.5. (1) Let ρ be a π – group congruence on S, then the kernel of ρ is a full N-subsemigroup of S and $\rho_{\ker\rho} \subset \rho$.

(2) If A is the intersection of all the full N-subsemigroups of S, the ρ_A is the smallest π—group congruence on S.

(3) For every $a \in$ S, $a\rho_A \in$ Reg (S/ρ_A) if and only if there exists $u \in A$ such that $u|a$.

(4) For every $a \in$ S, if $a\rho_A$ contains more than one element, then $a\rho_A \in$ Reg (S/ρ_A)

(5) Let A be a full, and closed N-subsemigroup of S. If there exists a $\in$ S such that for every u $\in$ A, $u|a$. Then there exists a π—group congruence on S such that $\rho_A \subset \rho$ and $\rho_A \neq \rho$.

Proof. (1) Since kerρ = $\{a \in$ S : $a\rho e$ for some $e \in$ E (S)$\}$ = $f\rho$ for every $f \in$ E (S). By theorem 2.1 and corollary 2.2, kerρ is a full N-subsemigroup and $\rho_{\ker\rho} \subset \rho$.

(2) It is the smallest π-group congruence, since A is the smallest full N-subsemigroup.

(3) Let $a\rho_A \in$ Reg $(S|\rho_A)$. If $a \in$ Reg S, then $e = a'a \in$ E (S) $\subset$ A, where $a' \in V(a)$. It follows that $a = ae$ and then $e|a$.

It $a \notin$ Reg S, let $n \in N$ be such that $a^n \in$ Reg S and $(a^n)' \in V(a^n)$, then

$a^n (a^n)' a \in$ Reg S and $a^{n-1} (a^n)' \in V(a^n(a^n)'a)$

Thus $a \neq a^n(a^n)'a$. Now since Reg $(S/\rho A)$ is a subgroup of S/ρ_A with identity $(a^n(a^n)'a)\rho_A$ *then* $a\rho_A = (a^n(a^n)'a)\rho_A$. By lemma 3.3, There exists $u \in A$, such that $u|a$.

Conversely, Suppose there exists $u \in A$ such that $u|a$, and $a = xuy$ for some x, y $\in S^1$. Let m, n be the positive integers such that y^m and a^n are regular, respectively. Suppose $(a^n)' \in V(a^n)$, and $(y^m)' \in V(y^m)$. It is easy to see $a^n (a^n)'a \in$ Reg S. Thus

$$
\begin{aligned}
a\rho_A &= (xuy)\rho_A = (xuy^{m'}\ (y^m)'y)\ \rho_A \qquad (\text{since } y^m\ (y^m)' \in E(S) \subset A)\\
&= (xuyy^{m-1}\ (y^m)'y)\rho_A\\
&= [xuy\ (y^{m-1}(y^m)'y)(a^{n-1}(a^n)^1 a)]\rho_A\\
&= (xuya^{n-1}(a^n)'a)\rho_A\\
&= (a^n\ (a^n)'a)\ \rho_A \in \text{Reg } (S/\rho_A)
\end{aligned}
$$

(4) Suppose $a \neq b$ and $a\rho_A b$, for $a, b \in S$. By lemma 3.3, there exist $u, v \in A$ such that $u|a$, $v|b$ and then $a\rho_A \in$ Reg (S/ρ_A) by (3). Equivalently, we know that if $a\rho_A$ is not a regular element in S/ρ_A, then $a\rho_A$ contains only one element.

(5) Recall that a subset A of an eventually regular semigroup is self-conjugate, if

$$aAa^{n-1}(a^n)' \subset A, \text{ and } a^{n-1}(a^n)'Aa \subset A$$

where n is the positive integer such that $a^n \in \text{Reg}S$, and $(a^n)' \in V(a^n)$.

Now, since $a^n(a^n)'$, $a(a^n)'a^{n-1} \in E(S) \subset A$. It follows that

$$a[(a^n)'a^n]a^{n-1}(a^n)' = a[(a^n)'a^{n-1}](a^n)(a^n)' \in A$$

and then $aAa^{n-1}(a^n)' \cap A \neq \emptyset$. so that $aAa^{n-1}(a^n)' \subset A$.

Similiarly, we have $a^{n-1}(a^n)'Aa \subset A$. Again since A is closed we obtain that

$$\rho = \{(x,y) \in S \times S : xu = vy \text{ for some } u,v \in A\}$$

is a group congruence on S by [5], and ker$\rho = A$. Thus $\rho A \subset \rho$. Since there exists $a \in S$, $u \mid a$ for every $u \in A$, and this implies $a\rho A \notin$ Reg $(S|\rho_A)$. Therefore $\rho_A \neq \rho$.

4. GROUP CONGRUENCE ON REGULAR SEMIGROUP

In this section, S will denote a regular semigroup. Let A be a full, self-conjugate and closed subsemigroup of S. In [3], D. R, Latorre showed that the relation defined by

$$\sigma_A = \{(a,b) \in S \times S : au = vb \text{ for some } u, v \in A\}$$

is a group congruence on S. Now, we will describe the group congruence on S by the full N-subsemigroup of S. First it is easy to show the following theorem.

Theorem 4.1. Let A be a full N-subsemigroup of S. Then ρ_A is a group congruence on S.

Theorem 4.2. Let A be a full N-subsemigroup of S. Then for any $a,b \in S$, the following conditions are equivalent.

(1) $a\rho_A b$;

(2) there exists $b_l \in V(b)$ such that $b_l a \in A$;

(3) there exists $e \in E(S)$ and $u \in A$ such that $ea = bu$

Proof. (1) implies (2). If $a\rho_A b$; then $(b_l a)\ \rho_A\ (b_l b)$ for all $b_l \in V(b)$ Then $b_l a \in A$ since $b_l b \in E(S) \subset A$ and A is a ρ_A-class.

(2) implies (3). Let $u = b_l a$, $e = bb_l$. Then $bu = bb_l a = ea$.

(3) implies (1). It is easy to see $a\rho_A = (ea)\rho_A = (bu)\rho_A = b\rho_A$ since ρ_A is a group congruence and $u \in A$.

Theorem 4.3. Let A be a full subsemigroup of S. Then the following conditions are equivalent.

(1) A is a N-subsemigroup.

(2) A is unitary and self-conjugate

(3) A is closed and self-congugate

(4) A is unitary and symmetric.

Proof. (1) implies (2). Let $a \in A$, $x \in S$ and $ax \in A$. Then $Ax \cap A \neq \emptyset$ and then $Ax \subset A$. Let $x_l \in V(x)$, then $xx_l \in E(S) \subset A$. So that $x = xx_l x \in Ax \subset A$. We also have $xa \in A$ implies $x \in A$. Thus A is unitary.

Let $x \in S$ and $x_l \in V(x)$. Then $x_l(xx_l)x = x_l x \in E(S) \subset A$ and thus $Ax \cap A \neq \emptyset$. So $x_l Ax \subset A$ which implies that A is self-conjugate.

(2) implies (3). It is obvious that $A \subset Aw$. Let $x \in Aw$, Then there exists $u \in A$ such that $ux \in A$ and then $x \in A$ since A is unitary. So $A = Aw$.

(3) implies (4). Let $x,y \in S$ be such that $xy \in A$. Suppose $x_l \in V(x)$, then $x_l xyx \in x_l Ax \subset A$, and then $yx \in A$ since $xx_l \in E(S) \subset A$ and A is closed. Thus A is symmetric.

Let $a \in A$, $x \in S$ and $ax \in A$, then $x \in A$ since A is closed. If $xa \in A$ then $ax \in A$ since A is symmetric. Thus $x \in A$, which implies that A is unitary.

(4) implies (1). $x, y \in S^1$ and $xAy \cap A \neq \emptyset$. Then there exists $a \in A$ such that $xay \in A$, and then $ayx \in A$ by the symmetry of A. So $yx \in A$, since A is unitary. For every $b \in A$, $by\ x \in A$ and then $xby \in A$. So $xAy \subset A$.

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