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COMMON FIXED POINTS FOR FUZZY MAPPINGS WITH THEIR ASSOCIATED MULTIMAPPINGS

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ABSTRACT : In this paper we obtain a result on fixed points of fuzzy mappings with their associated multimappings which extends the result of [15] and [3].

1. INTRODUCTION

Several fixed point theorems for fuzzy mappings have been obtained by researchers [1, 2, 4-7, 9-14]. Heilpern [7] obtained a fixed point theorem for contractive type fuzzy mappings in metric space. Also Lee and Cho [9] studied fixed point theorems for contractive type fuzzy mappings which are fuzzy analogue of fixed point theorems for contractive type multivalued mappings (see [8]). Lee et al. [10-13] discussed common fixed points of a sequence of fuzzy mappings, especially they [13] showed existence of common fixed points for a pair of fuzzy mappings.

2. PRELIMINARIES

We state some useful notations, definitions and results.

Definition 2.1. Let (X,d) be any metric linear space. A fuzzy set in X is a function with domain X and values in [0,1].

Definition 2.2. If A is a fuzzy set and x ∈ X, the function values A(x) (or μA(x) is called the grade of membership of x in A.

Definition 2.3. The α-level set of a fuzzy set A, denoted by

$A_\alpha = \{x : A(x) \geq \alpha\}$ if $\alpha \in (0,1)$ and $A_0 = \{x : A(x) > 0\}$.

Definition 2.4. A fuzzy set A is said to be an approximate quantity iff A_α is compact and convex in X for each $\alpha \in [0,1]$ and $\sup_{x\in X} A(x) = 1$.

We denote by W(X) the sub collection of approximate quantities, C(X) the set of compact subsets of X, (C(X),H) the Hausdorff metric space and $D(A,B) = \inf_{x\in A,\, y\in B} d(x,y)$.

Definition 2.5. Let X be an arbitrary set and Y be any metric linear space. F is said to be a fuzzy mapping iff F is a mapping from the set X into W(Y) i.e. $F(x) \in W(Y)$ for each $x \in X$.

Definition 2.6. [16] A point $p \in X$ is called a fixed point of a fuzzy mapping $F : X \to W(X)$ if $F_p(P) \geq F_p(x)$, for all $x \in X$.

Definition 2.7. [16] If $F : X \to W(X)$ be a fuzzy mapping. Then, an associated multi mapping $F^\wedge : X \to CB(X)$ is defined by

$$F^\wedge(x) = \{y \in X : F_x(y) = \max_{u \in X} F_x(u)\}$$

Definition 2.8. [7] $D_\alpha(A,B) = \inf_{x \in A_\alpha\ y \in B_\alpha} d(x,y)$

$$H_\alpha(A,B) = \underset{\alpha}{dist}(A_\alpha, B_\alpha);$$

$$D(A,B) = \sup_\alpha H_\alpha(A,B);$$

The following Lemma is due to Heilpern.

Lemma 2.9. [7] If $\{x_0\} \subset A$, then $D_\alpha(x_0,B) \leq H_\alpha(A,B)$ for each $B \in W(X)$.

Lemma 2.10. $F_p^\wedge(p) \geq F_p^\wedge(x)$ iff $p \in F^\wedge(p)$ for all $x \in X$.

Ray [15] proved the following.

Theorem 2.11. Let (X,d) be a complete metric space, R^+ the set of all non negative real numbers and $w : R^+ \to R^+$ is a continuous function such that $0 < w(r) < r$, for all r in $R^+ - \{0\}$. Then self mappings f, g and h of X have a unique common fixed point if

(i) $d(fx,gy) \leq d(hx,hy) - w(d(hx,hy))$,

(ii) h is continuous,

(iii) $f(X) \cup g(X) \subseteq h(X)$.

Change [3] proved the following theorem.

Theorem 2.12. Let $F, G : X \to W(X)$ be two fuzzy mappings and $F^\wedge, G^\wedge$ be their associated multimappings respectively. Suppose that for any $x,y \in X$, the following holds

$$H(F^\wedge(x), G^\wedge(y)) \leq \phi(d(x,y), d(x,F^\wedge(x)), d(x,G^\wedge(y)), d(y,F^\wedge(y))\})$$

where the function ϕ satisfies the following conditions

(i) $\phi : [0,\infty)^5 \to [0,\infty)$ is non decreasing for each variable and ϕ is upper semi continuous,

(ii) ϕ (t,t,t,at,bt) $\leq \phi$ (t), for all $t \geq 0$, where ϕ (t) is a function from $[0,\infty)^5$ into $[0,\infty)$ such that $\phi(t) < t$ for all $t > 0$, $\phi(0) = 0$; a, b = 0, 1, 2; a + b =2.

Let $\beta > 1$, $x_0 \in X$, $x_1 \in F^\wedge (x_0)$ and define a non-negative real sequence as following ;

$t_{k+1} = t_k + \phi(\beta(t_k - t_{k-1}))$; k = 1, 2, ...; $t_0 = 0$; $t_1 > d(x_0,x_1)$.

If (t_k) converges, then $F^\wedge$ and $G^\wedge$ have a unique common fixed point.

3. MAIN RESULT

We prove the following.

Theorem 3.1. Let (X,d) be a complete metric space. Let F,G : X $\to$ W(X) be two fuzzy mappings and $F^\wedge$, $G^\wedge$ be their associated multimappings defined from X into C(X) (the set of compact subsets of X) satisfying

$$\text{(i)}\quad H^2(F^\wedge x, G^\wedge Y) \leq \max\{D^2(x,F^\wedge x), D^2(y,G^\wedge y), D(x,F^\wedge x)D(y,G^\wedge y)$$

$$(\tfrac{1}{2})d(x,y)[D(x,G^\wedge y)+D(y,F^\wedge x)]\}$$

$$-w(\max\{D^2(x,F^\wedge x), D^2(y,G^\wedge y), D(x,F^\wedge x)D(y,G^\wedge y),$$

$$(\tfrac{1}{2})d(x,y)[D(x,G^\wedge y)+D(y,F^\wedge x)]\}$$

for all x,y $\in$ X ; w : $R^+ \to R^+$ a non-decreasing continuous function such that $0 < w(r) < r$, for all $r > 0$ and $w(0) = 0$.

Then there exists a common fixed point of $F^\wedge$ and $G^\wedge$. Also F and G have a common fixed point.

Proof. Let $x_0 \in X$. Since C(X) is compact. So we can construct a sequence $\{x_n\}$ such that $x_{2n+1} \in F^\wedge x_{2n}$ and $x_{2n+2} \in G^\wedge x_{2n+1}$ $\forall$n= 0,1,2 ... with

$d(x_{2n}, x_{2n-1}) = D(x_{2n}, F^\wedge x_{2n})$ and $d(x_{2n+1}, x_{2n+2}) = D(x_{2n+1}, G^\wedge x_{2n+1})$ and

$d(x_{2n+1}, x_{2n+2}) \leq H(F^\wedge x_{2n}, G^\wedge x_{2n+1})$

Using inequality (i), we have

$$d^2(x_1,x_2) \leq H^2(F^\wedge x_0, G^\wedge x_1)$$

$$\leq \max\{D^2(x_0,F^\wedge x_0), D^2(x_1,G^\wedge x_1), D(x_0,F^\wedge x_0)D(x_1,G^\wedge x_1),$$

$$(\tfrac{1}{2})d(x_0,x_1)[D(x_0,G^\wedge x_1)+D(x_1,F^\wedge x_0)]\}$$

$$-w(\max\{D^2(x_0,F^\wedge x_0),D^2(x_1,G^\wedge x_1),D(x_0,F^\wedge x_0)D(x_1,G^\wedge x_1),$$

$$(½)d(x_0,x_1)[D(x_0,G^\wedge x_1)+D(x_1,F^\wedge x_0)]\}$$

$$=\max\{d^2(x_0,x_1),d^2(x_1,x_2),d(x_0,x_1)d(x_1,x_2),(½)d(x_0,x_1)d(x_0,x_2)\}$$

$$-w(\max\{d^2(x_0,x_1),d^2(x_1,x_2),d(x_0,x_1)d(x_1,x_2),(½)d(x_0,x_1)d(x_0,x_2)\})$$

$$\le\max\{d^2(x_0,x_1),d^2(x_1,x_2),d(x_0,x_1)d(x_1,x_2),$$

$$(½)d(x_0,x_1)[d(x_0,x_1)+d(x_1,x_2)\}$$

$$-w(\max\{d^2(x_0,x_1),d^2(x_1,x_2),d(x_0,x_1)d(x_1,x_2),$$

$$(½)d(x_0,x_1)[d(x_0,x_1)+d(x_1,x_2)]\}) \quad \ldots (1)$$

If $d(x_1,x_2) > d(x_0,x_1)$, then we have

$d^2(x_1,x_2) \le d^2(x_1,x_2) - w(d^2(x_1,x_2))$,

leading to a contradiction. Therefore, $d(x_1,x_2) \le d(x_0,x_1)$ and by (1)

$$d^2(x_1,x_2) \le d^2(x_0,x_1) - w(d^2(x_0,x_1)) \quad \ldots (2)$$

Further using (i), we have

$$d^2(x_2,x_3) \le H^2(G^\wedge x_1, F^\wedge x_2)$$

$$= H^2(F^\wedge x_2, G^\wedge x_1)$$

$$\le\max\{D^2(x_2,F^\wedge x_2),D^2(x_1,G^\wedge x_1),D(x_2,F^\wedge x_2)D(x_1,G^\wedge x_1)$$

$$(½)d(x_2,x_1)[D(x_2,G^\wedge x_1)+D(x_1,F^\wedge x_2)\}$$

$$-w(\max\{D^2(x_2,F^\wedge x_2),D^2(x_1,G^\wedge x_1),D(x_2,F^\wedge x_2)D(x_1,G^\wedge x_1)$$

$$(½)d(x_2,x_1)[D(x_2,G^\wedge x_1)+D(x_1,F^\wedge x_2)\})$$

$$=\max\{d^2(x_2,x_3),d^2(x_1,x_2),d(x_2,x_3)d(x_1,x_2),$$

$$(½)d(x_1,x_2)d(x_1,x_3)\}$$

$$-w(\max\{d^2(x_2,x_3),d^2(x_1,x_2),d(x_2,x_3)d(x_1,x_2),$$

$$(½)d(x_1,x_2)d(x_1,x_3)\}).$$

This implies with same arguments as above

$$d^2(x_2,x_3) \le d^2(x_1,x_2) - w(d^2(x_1,x_2)) \quad \ldots (3)$$

Similarly,

$d^2(x_3,x_4) \leq d^2(x_2,x_3) - w(d^2(x_2,x_3))$ ... (4)

$d^2(x_n,x_{n+1}) \leq d^2(x_{n-1},x_n) - w(d^2(x_{n-1},x_n))$ ... (3)

Adding (2) to (5), we have

$$\sum_{i=0}^{n-1} w(d^2(x_i,x_{i+1}) \leq d^2(x_0,x_1) - d^2(x_n,x_{n+1}) \leq d^2(x_0,x_1)$$

This implies, $\sum_{i=0}^{n-1} w(d^2(x_i,x_{i+1}) < \infty$ and $\lim w(d^2(x_n,x_{n+1})) = 0$ ... (6)

Since $\{d^2(x_n,x_{n+1})$ is a decreasing sequence of non-negative terms, therefore (6) implies that $\lim_{n\to\infty} d^2(x_n,x_{n+1}) = 0$ and so $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$

Suppose that $\{x_n\}$ is not a Cauchy sequence, then there is an $\varepsilon > 0$ such that for each even integer 2K, there are even integers $2m > 2n > 2K$ such that $d(x_{2m}.x_{2n}) > \varepsilon$. ... (7)

Also for each even integer 2k, we can find the least even integer 2m, exceeding 2n such that

$d(x_{2n}, x_{2m-2}) \leq \varepsilon$ ... (8)

then $\varepsilon < d(x_{2m},x_{2n}) \leq d(x_{2n},x_{2m-2}) + d(x_{2m-2},x_{2m-1}) + d(x_{2m-1}, x_{2m})$

This implies $d(x_{2m},x_{2n}) \to \varepsilon$ as $k \to \infty$.

Using triangular property of metric space, we have

$| d(x_{2m},x_{2n+1}) - d(x_{2m},x_{2n}) | \leq d(x_{2n},x_{2n+1})$,

$| d(x_{2m+1},x_{2n+1}) - d(x_{2m},x_{2n+1}) | \leq d(x_{2m},x_{2m+1})$,

$| d(x_{2m},x_{2n+2}) - d(x_{2m},x_{2n+1}) | \leq d(x_{2n+1},x_{2n+2})$,

and $| d(x_{2m},x_{2n+2}) - d(x_{2m+1},x_{2n+1}) | \leq d(x_{2n+1},x_{2n+2})$

which implies on letting $\lim k \to \infty$,

$d(x_{2m},x_{2m+1}) \to \varepsilon$, $d(x_{2m+1},x_{2n+1}) \to \varepsilon$, $d(x_{2m},x_{2n+2}) \to \varepsilon$

and $d(x_{2m+1},x_{2n+2}) \to \varepsilon$.

Now using (i), we have

$$d^2(x_{2m+1},x_{2n+2}) \leq H^2(F^\wedge x_{2m}, G^\wedge x_{2n+1})$$

$$\le \max\{D^2(x_{2m}, F^\wedge x_{2m}), D^2(x_{2n+1}, G^\wedge x_{2n-1}),$$

$$D(x_{2m}, F^\wedge x_{2m})D(x_{2n+1}, G^\wedge x_{2n+1}),$$

$$(½)d(x_{2m}, x_{2n})[D(x_{2m}, G^\wedge x_{2n+1}) + D(x_{2n+1}, F^\wedge x_{2m})]\}$$

$$-w \max\{D^2(x_{2m}, F^\wedge x_{2m}), D^2(x_{2n+1}, G^\wedge x_{2n+1}),$$

$$D(x_{2m}, F^\wedge x_{2m})D(x_{2n+1}, G^\wedge x_{2n+1}),$$

$$(½)d(x_{2m}, x_{2n})[D(x_{2m}, G^\wedge x_{2n+1}) + D(x_{2n+1}, F^\wedge x_{2m})]\}$$

or

$$d^2(x_{2m+1}, x_{2n+2}) \le \max\{d^2(x_{2m}, x_{2m+1}), d^2(x_{2n+1}, x_{2n+2}),$$

$$d(x_{2m}, x_{2m-1})d(x_{2n+1}, x_{2n+2}),$$

$$(½)d(x_{2m}, x_{2n})[d(x_{2m}, x_{2n+2}) + d(x_{2n+1}, x_{2m+1})]\}$$

$$-w(\max\{d^2(x_{2m}, x_{2m+1}), d^2(x_{2n+1}, x_{2n+2}),$$

$$d(x_{2m}, x_{2m+1})d(x_{2n+1}, x_{2n+2}),$$

$$(½)d(x_{2m}, x_{2n})[d(x_{2m}, x_{2n+2}) + d(x_{2n+1}, x_{2m+1})]\}$$

which implies on letting $k \to \infty$,

$$\varepsilon^2 \le \varepsilon^2 - w(\varepsilon^2)$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, so $\{x_n\}$ converges to a point $u \in X$. Thus, we have

$$D^2(x_{2n+1}, G^\wedge u) \le H^2(F^\wedge x_{2n}, G^\wedge u)$$

$$\le \max\{D^2(x_{2n}, F^\wedge x_{2n}), D^2(u, G^\wedge u), D(x_{2n}, F^\wedge x_{2n})D(u, G^\wedge u),$$

$$(½)d(x_{2n}, u)[D(x_{2n}, G^\wedge u) + D(u, F^\wedge x_{2n})]\}$$

$$-w(\max\{D^2(x_{2n}, F^\wedge x_{2n}), D^2(u, G^\wedge u), D(x_{2n}, F^\wedge x_{2n})D(u, G^\wedge u),$$

$$(½)d(x_{2n}, u)[D(x_{2n}, G^\wedge u) + D(u, F^\wedge x_{2n})]\}$$

$$\le \max\{d^2(x_{2n}, x_{2n+1}), D^2(u, G^\wedge u), d(x_{2n}, x_{2n+1})D(u, G^\wedge u),$$

$$(½)d(x_{2n}, u)[D(x_{2n}, G^\wedge u) + D(u, F^2 x_{2n})]\}).$$

Taking limit $n \to \infty$, we have

$$D^2(u,G^\wedge u) \le \max\{0, D^2(u,G^\wedge u), 0, 0\} - w(\max\{0, D^2(u,G^2 u), 0, 0\})$$

$$D^2(u,G^\wedge u) \le D^2(u,G^2 u) - w(D^2(u,G^2 u))$$

which implies that $u \in G^\wedge u$.

Similarly, we can prove that $u \in F^\wedge u$. Hence $u \in F^\wedge u \cap G^\wedge u$.

Using Lemma 2.10, it is obvious that $u \in Fu \cap Gu$.

Corollary 3.2. Let $\{F_i\}$ be a sequence of fuzzy mappings of X to W (X) and $\{F^\wedge_i\}$ the sequence of its associated multimappings from X to C(X).

Suppose, for any positive integers i, j ; i ≠ j and x,y ∈ X, following condition holds.

$$H^2(F_i^\wedge x, F_j^\wedge y) \le \max\{D^2(x,F_i^\wedge x), D^2(y,F_j^\wedge y), D(x,F_i^\wedge x)D(y,F_j^\wedge y),$$
$$(\tfrac{1}{2})d(x,y)[D(x,F_j^\wedge y) + D(y,F_i^\wedge x)]\}$$
$$-w(\max\{D^2(x,F_i^\wedge x), D^2(y,F_j^\wedge y), D(x,F_i^\wedge x)D(y,F_j^\wedge y),$$
$$(\tfrac{1}{2})d(x,y)[D(x,F_j^\wedge y) + D(y,F_i^\wedge x)]\}$$

where w : $R^+ \to R^+$ is a non-decreasing continuous function such that 0 < W (r) < r, for all r > 0 and w (0) = 0. Then there exists a fixed point of $\{F_i^\wedge\}$.

Corollary 3.5. Let (X, d) be complete metric space and $\{F_i\}$, $\{G_i\}$ the sequence of fuzzy mapping of X to W(X) and $\{F_i^\wedge\}, \{G_i^\wedge\}$ be the sequences of their associated multi-mapping of X to C(X) (the set of compact subsets of X) converging pointwise to the associated multimappings $F^\wedge$, $G^\wedge$ of fuzzy mapping F and G respectively, Satisfying

$$H^2(F_n^\wedge x, G_n^\wedge x) \le \max\{D^2(x,F_n^\wedge x), D^2(y,G_n^\wedge y), D(x,F_n^\wedge x)D(y,G_n^\wedge y),$$
$$(\tfrac{1}{2})d(x,y)[D(x,G_n^\wedge y) + D(y,F_n^\wedge x)]\}$$
$$-w(\max\{D^2(x,F_n^\wedge x), D^2(y,G_n^\wedge y), D(x,F_n^\wedge x)D(y,G_n^\wedge y),$$
$$(\tfrac{1}{2})d(x,y)[D(x,G_n^\wedge y) + D(y,F_n^\wedge x)]\})$$

Then $F^\wedge$ and $G^\wedge$ have a unique common fixed point.

Proof. Let $x_0 \in$ X. Define the sequence $\{x_n\}$ with

$x_{2n+1} \in F_n^\wedge x_{2n}$ and $x_{2n+2} \in G_n^\wedge x_{2n+1}$ $\forall n=0,1,2,\ldots\ldots$

such that $d(x_{2n+1,}x_{2n+2}) \leq H(F_n^{\wedge}x_{2n,}G_n^{\wedge}x_{2n+1})$.

Now, for x,y $\in$ X, we have

$$|D(y,F_n^{\wedge}x) - D(y,F^{\wedge}x)| \leq H(F_n^{\wedge}x,F^{\wedge}x) ;$$

$$|D(y,G_n^{\wedge}y) - D(y,G^{\wedge}y)| \leq H(G_n^{\wedge}y,G^{\wedge}y)$$

$$|D(x,F_n^{\wedge}x) - D(x,F^{\wedge}x)| \leq H(F_n^{\wedge}x,F^{\wedge}x) \text{ and}$$

$$|D(x,G_n^{\wedge}y) - D(x,G^{\wedge}y)| \leq H(G_n^{\wedge}y,G^{\wedge}y).$$

Now, since H is continuous and $\{F_n^{\wedge}\},\{G_n^{\wedge}\}$ cnverge pointwise to $F^{\wedge}$ and $G^{\wedge}$ respectively, hence we get

$$H^2(F^{\wedge}x,G^{\wedge}y) \leq \max\{D^2(x,F^{\wedge}x),D^2(y,G^{\wedge}y),D(x,F^{\wedge}x)D(y,G^{\wedge}y),$$

$$(\tfrac{1}{2})d(x,y)[D(x,G^{\wedge}y)+D(y,F^{\wedge}x)]\}$$

$$-w(\max\{D^2(x,F^{\wedge}x),D^2(y,G^{\wedge}y),D(x,F^{\wedge}x)D(y,G^{\wedge}y),$$

$$(\tfrac{1}{2})d(x,y)[D(x,G^{\wedge}y)+D(y,F^{\wedge}x)]\}$$

Now rest part of the proof is similar to the proof of the theorem 2.11.

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