Jour. Pure Math., Vol. 19, 2002, pp. 81-85

WEAK FINITE OPERATORS

SALAH MECHERI

ABSTRACT : Let $L(H)$ denote the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space H into itself. For $A \in L(H)$, we define the derivation $\delta_A : L(H) \to L(H)$ by $\delta_A(X) = AX - XA$. In this paper we introduce the notion of weak finite operators for which we give characterization and we prove that this class of operators is norm dense in $L(H)$ by generalizing H. Yang and S. N. El alami's results [8], [25].

2000 Mathematics Subject Classification : 47A30, 47B10, 47B47.

Key words : Derivation, Weak finite operators.

1. INTRODUCTION

Let $L(H)$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space H. The generalized derivation operator $\delta_{A,B}$ associated with (A,B), defined on $L(H)$ by

$$\delta_{A,B}(X) = AX - XB$$

was initially studied by M. Rosenblum [13]. The properties of such operators have been much studied (see for example [1], [2], [4], [12], [13] and [14]).

If $A = B$, $\delta_{A,A} = \delta_A : L(H) \to L(H)$ is called the inner derivation defined by

$$\delta_A(X) = AX - XA.$$

The theory of derivations has been extensively dealt with in the literature (see, [5], [8], [9], [11], [16], [17], [18] and [19]. Let $\mathcal{N}$ be the set $\{A \in L(H) : I \notin \overline{R(\delta_A)}\}$. In [15] we gave some operators for which $I \notin \overline{R(\delta_A)}$. In [25] H. Yang, shows that the set $\mathcal{N}$ is norm-dense in $L(H)$. Also S. N. Elalami [8], shows that the set $\mathcal{M}_w = \{A \in L(H) : I \notin \overline{R(\delta_A)}^w\}$, is norm-dense in $L(H)$, where $\overline{R(\delta_A)}^w$ denotes the weak closure of the range of δ_A. In order

Supported by the Research Centre Project No. **Math/1422/10**

to generalize these results we prove that the set $\mathcal{T}_w = \left\{A \in L(H) : \forall K \in \mathcal{K}(H), I \notin \overline{R(\delta_A)}^w\right\}$ is also norm-dense in $L(H)$, where $\mathcal{K}(H)$ is the ideal of all compact operators.

Definition 1.1. *We shall say that a certain property (P) (of operators acting on a Hilbert space) is a bad property, written b-property.*

(i) If A has the Property (P) and T similar to A, then $\alpha + \beta A$ has the property (P), for all $\alpha \in \mathbb{C}$ and $\beta \neq 0$,

(ii) If A has the property (P) and T similar to A, then T has the property (P), and

(iii) If A has the property (P) and $\sigma(A) \cap \sigma(B) \neq \phi$, then $A \oplus B$ has the property (P).

Example of bad properties are frequent in the literature ; namely, (1) T is not cyclic, (2) the spectrum of T is disconnected (or $\sigma(T)$) has infinitely many components, or c components, where c is the power of the continuim), (3) $\sigma(T)$ has non empty interior, (4) the commutant of T is not algebraic, etc., are examples of properties satisfying (i), (ii), (iii).

Definition 1.2. *An operator $A \in L(H)$ is called weak finite, if $I \notin \overline{R(\delta_A)}^w$.*

Theorem 1.3. *The set $\mathcal{T}_w = \left\{A \in L(H) : \forall K \in \mathcal{K}(H), I + K \notin \overline{R(\delta_A)}^w\right\}$ is norm-dense in $L(H)$.*

Proof. By using [10], Theorem 3.5.1, it suffices to prove that the property $A \in \mathcal{T}w$ is a b-property. It is easy to see that $R(\delta_A) = R(\delta_{\alpha A+\beta})$ for $\alpha \in \mathbb{C}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$ and $X \in L(H)$; hence if $A \in \mathcal{T}_w$, then $\alpha A + \beta \in \mathcal{T}_w$. Now if $S \in L(H)$ and S is invertible, then for all $X \in L(H)$,

$$S(AX - XA)S^{-1} = (SAS^{-1})(SXS^{-1}) - (SXS^{-1})(SAS^{-1}).$$

Thus $S\overline{R(\delta_A)}^w S^{-1} = \overline{R(\delta_{SAS^{-1}})}^w$. Hence if $I + K \in \overline{R(\delta_A)}^w$, then $I + SKS^{-1} \in \overline{R(\delta_{SAS^{-1}})}^w$. It follows by the above argument that if $\overline{R(\delta_A)}^w$ contains $I + K$, then it is also true for all operator similar to A. Hence $A \in \mathcal{T}_w$ is invariant for similarity.

Let $\varepsilon = H \oplus H$ and $B = A \oplus C$. Suppose that there exists $\{X_\alpha\} \in L(\varepsilon)$ such that $[(A \oplus C) X_\alpha - X_\alpha(A \oplus C) \xrightarrow{w} I_\varepsilon \oplus K$. Let P_0 be the orthogonal projection on H, K_1 denotes the compression of K to H, i.e., $K_1 = P_0KP_0|H$ and X_{α_1} denotes the compression of X_α to H. Then $AX_{\alpha_1} - X_{\alpha_1}A \xrightarrow{w} I_H \oplus K_1$. So, if $A \oplus C \notin \mathcal{T}_w$ then $A \notin \mathcal{T}_w$. Note that the hypothesis on the spectrum of C can be dropped here. ∎

2. EXAMPLES OF WEAK FINITE OPERATORS

Theorem 2.1. *Every operator $A \in L(H)$ which has a pole of order v, is weak finite.*

Proof. The operator A can be decomposed as $A = B \oplus C$ on $H = R(P_\lambda) \oplus G$, where $(B-\lambda)^v = 0$ on $R(P_\lambda)$ and P_λ is the Riesz projection. Since B is algebraic, $I \notin \overline{R(\delta_B)}^w$. Hence $I \notin \overline{R(\delta_A)}^w$. ∎

Lemma 2.2. *If $H = \oplus_{i=1}^n H_i$ where $\dim H_n < \infty$ (orthogonal direct sum), and if $A = \oplus_{i=1}^n A_i$ on H, then every operator in $\overline{R(\delta_A)}^w$ vanishes. Consequently*

$$\overline{R(\delta_A)}^w \cap \{A^*\}' = \{0\}.$$

Proof. *Suppose that*

$$AX_\alpha - X_\alpha A \xrightarrow{w} P,$$

where P is a positive operator and $\{X_\alpha\}$ a generalized sequence in $L(H)$. Noting by Q_n the orthogonal projection on H_n, we get

$$AX_\alpha^{(n)} - X_\alpha^{(n)} A \xrightarrow{w} Q_n(P \mid H_n),$$

where $X_\alpha^{(n)}$ is the compression of X_α to H_n. *Since $\dim H_n < \infty$, $Q_n(P \mid H_n) \in R(\delta_{A_n})$ this implies that $Q_n(P \mid H_n) = 0$ since this last operator is positive on H_n. Then*

$$0 = Q_n P Q_n = \left(\sqrt{P}Q_n\right)^* \left(\sqrt{P}Q_n\right)$$

from where $\sqrt{P}Q_n = 0$ or $PQ_n = 0$. As $\sum_{i=1}^n Q_i = I$, it follows that $P = 0$.

For the consequence, it suffices to remark that if

$$T \in \overline{R(\delta_A)}^w \cap \{A^*\}',$$

then $T^* T \in \overline{R(\delta_A)}^w$. ∎

Lemma 2.3. *Let $A \in L(H)$. If A is normal and countable, then every positive operator in $\overline{R(\delta_A)}^w$ vanishes. Consequently*

$$\overline{R(\delta_A)}^w \cap \{A\}' = \{0\}.$$

Proof. *Let T be a positive operator. In the decomposition of*

$H = \overline{R(A)} \oplus \ker A$,

We can write

$$A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}; \quad T = \begin{bmatrix} R & * \\ * & S \end{bmatrix}.$$

If $T \in \overline{R(\delta_A)}^w$, *then* $S = 0$ *and* $R \in \overline{R(\delta_A)}^w$. *Since* $\overline{R(A)}$ *is a countable orthogonal basis composed of eingenvectors of A. Also* $R = 0$ *by Lemma 2.2. Since T is positive, it results that* $T = 0$.

Concerning the consequence it suffieces to remark that $\{A\}' = \{A^*\}'$.

Theorem 2.4. *Let* $A \in L(H)$. *If A has a countable spectrum and if P(A) is normal for certain non-constant polynomial P, then A is weak finite.*

Proof. It follows from [15, Lemma 3] that we can assume A has no poles. Now suppose that

$$AX_\alpha - X_\alpha A \xrightarrow{w} I.$$

It is easy to see that

$$P(A)X_\alpha - X_\alpha P(A) \xrightarrow{w} P'(A),$$

which implies that

$$P'(A) \in \overline{R(\delta_{P(A)})}^w \cap \{P(A)\}'.$$

It follows from Lemma 2.3 that $P'(A) = 0$ and by the theorem of the minimum equation [7], P' vanishes on some neighborhood of $\sigma(A)$. Hence P is constant, which is absurd. ∎

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Department of Mathematics
King Saud University College of Science
P.O. Box 2455, Riyadh 11451
Saudi Arabia