Jour. Pure Math., Vol. 19, 2002, pp. 49-59

INTEGRATION OF A MORE GENERAL CLASS OF FUNCTION ON A TOPOLOGICAL SPACE

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ABSTRACT : Applying the concepts of regular θ-Baire measure [2] and regular θ-Borel measure [3], we here introduce the concept of integration of θ-Baire functions [2] and θ-Borel functions [3] with H-set support and finally we give an analogue but a generalized version of the famous Riesz-Markoff represantation theorem.

2000 AMS Subject Classification : 28A25

Key words and phrases : Locally θ-H-closed space [5], θ-CR-space [5], θ-T_2-space [5], θ-open set [6], θ-closed set [6], H-set [6], θ-Baire set [2], θ-Borel set [3], Continuous function with H-set support [2], Simple θ-Baire function [2], θ-Baire measure [2], θ-Borel measure [3], θ-content [3], regular θ-content [3].

1. INTRODUCTION

The theory of integration can be developed on two types of topological spaces, viz., locally compact T_2-space and complete metric space. We know that every continuous function with compact support is integrable on a locally compact T_2-space with respect to a regular Baire measure or regular Boarel measure. In [5] a class of topological spaces is innovated viz., locally θ-H closed, θ-CR, θ-T_2 space which is more general than a locally compact T_2-space in a non-regular space ; also in [3], it is shown that on a locally θ-H-closed, θ-CR, θ-T_2 space, there always exists a regular θ-Borel measure. Using all these concepts, we here introduce the concept of integration of a continuous function with H-set support with respect to a regular θ-Baire measure or θ-Borel measure on a locally θ-H-closed, θ-CR, θ-T_2 space as is done by first integrating characteristic functions and subsequently, simple functions, non-negative *-measurable functions, then *-measurable functions in general (*—denoting either Baire or Borel).

Theorem 4.13. [3], which ascertain the possibility of extending a θ-Baire measure ν on a locally θ-H-closed, θ-CR, θ-T_2 space X to a unique regular θ-Borel measure μ might

* The Research is supported by a Minor Research Project of the UGC, India.

however arouse an atom of ambiguity which requires a solution at the very outset for the sake of consistency and this shall be done in Theorem 1.1 of next article.

However throughout the whole paper we would be primarily interested in the relationship between a regular θ-Borel measure and the integral which it assigns to continuous functions with H-set support.

Unless otherwise stated X shall denote a locally θ-H-closed, θ-CR, θ-T_2 space and $\mathcal{H}$ shall denote the class of all continuous functions on X with H-set support.

PREREQUISITES

0.1. Let A be an H-set and U be a θ-open set in X with $A \subset U$ ensure the existence of some $f \in \mathcal{H}$ such that $f = 0$ on $X \setminus U$, $f = 1$ on A and $0 \leq f \leq 1$ [2].

0.2. The class of all θ-Baire functions is the smallest class of real-valued continuous functions on X which contains $\mathcal{H}$ and is closed under sequential pointwise limit [2].

0.3. If f is integrable, g is measureable and $f = g$ a.e. then g is also integrable and $\int f d\mu = \int g d\mu$[4].

0.4. If f is measurable, g is integrable and $|f| \leq |g|$ a.e., then f is integrable [4].

1. INTEGRATION OF CONTINUOUS FUNCTIONS WITH H-SET SUPPORT

Theorem 1.1. *Let μ be a θ-Borel measure on X, and ν the θ-Baire restriction of μ. If f is a θ-Baire function, then f is μ-integrable if and only if it is ν-integrable, and in this case* $\int f d\mu = \int f d\nu$.

Proof. As usual, there shall be no loss of generality if we consider $f \geq 0$; with the usual practice we can select a sequence of simple θ-Baire functions such that $0 \leq f_n \uparrow f$ ......(1); of course, they are θ-Borel functions too. If $N(f)$ denotes the set $\{x \in X : f(x) \neq 0\}$ then evidently $\mu(N(f_n)) = \nu(N(f_n))$ and it is clear that f_n is μ-integrable if and only if it is ν-integrable, and that in this case we have $\int f_n d\mu = \int f_n d\nu$; in view of (1), it is now assured, by simple measure-theoretic arguments that f is μ-integrable if and only if it is ν-integrable and that in this case the values of the two integrals are equal.

Example 1.2. Let X be the real line and if τ_1 is the Euclidean topology on X and τ_2 is the topology of countable complements on X, we define τ to be the smallest topology generated by $\tau_1 \cup \tau_2$. Let $[a,b]$ be any closed subset of (X, τ_1). Then [a,b] is a compact subset of (X,τ_1), also [a,b] is an H-set but not compact subset of (X,τ). Hence any real valued continuous function with support $[a,b]$ in (X,τ_1) is a continuous function with H-set support $[a,b]$ in (X,τ) but not with compact support.

Note 1.3. The class $\mathcal{H}$ of real valued continuous functions on X with H-set support is wider than the class of real valued continuous functions on X with with compact set support if X is non-regular.

Theorem 1.4. *If f be a real valued continuous function with H-set support on X, then f is ν-integrable, where ν is the θ-Baire measure.*

Proof. Let C be the H-set such that $f = 0$ on $X \setminus C$. First we prove that f is bounded on C. Infact, $f(C)$ is an H-set in $\mathbb{R}$, Since $\mathbb{R}$ is regular $f(C)$ is compact set in $\mathbb{R}$. Thus $f(C)$ is a bounded set in $\mathbb{R}$ and so f is a bounded function. Let α be an upper bound of f on C, then clearly $|f(x)| \leq \alpha\chi_c$; also $\alpha\chi_c$ is integrable and f is a θ-Baire function [2] hence f is ν-measurable [2], so f is ν-integrable on X (by 0.4). This completes the theorem.

Note 1.5. It follows from Theorem 1.4 that if μ is a θ-Borel measure, and ν is the θ-Baire restriction of μ, then every continuous function with H-set support is ν-integrable and hence μ-integrable by Theorem 1.1.

With $\mathcal{H}$—denoting the class of continuous functions with H-set support we can now assure a property common to θ-Baire measure and regular θ-Borel measure. Infact, if any two θ-Baire measure assign the same integral to each f in H, then they are identical ; the same is true for two θ-Borel measures as well.

Theorem 1.6. *If ν_1 and ν_2 are θ-Baire measure on X such that $\int f d\nu_1 = \int f d\nu_2$ for every f in $\mathcal{H}$, then $\nu_1 = \nu_2$.*

Proof. Let D be any arbitrary $H\text{-}G_\delta^\theta$ set, then χ_D is a simple θ-Baire function and there exists a sequence of functions $\{f_n\}$ in $\mathcal{H}$ such that $f_n \downarrow \chi_D$ [2], this implies that $\int f_n d\nu \downarrow \nu(D)$, for every θ-Baire measure ν. Hence $\int f_n d\nu_1 \downarrow \nu_1(D)$ and $\int f_n d\nu_2 \downarrow \nu_2(D)$. But, according to assumption $\int f_n d\nu_1 = \int f_n d\nu_2$, for every n. So $\nu_1(D) = \nu_2(D)$ for every $H\text{-}G_\delta^\theta$ set i.e., $\nu_1 = \nu_2$ [2].

Theorem 1.7. *If μ_1 and μ_2 are regular θ-Borel measures on X such that $\int f d\mu_1 = \int f d\mu_2$ for every f in $\mathcal{H}$, then $\mu_1 = \mu_2$.*

Proof. Let ν_i be the θ-Baire restriction of μ_i, $i = 1, 2$; by Theorems 1.1 and 1.4 we have $\int f d\nu_1 = \int f d\nu_2$ for every f in $\mathcal{H}$; hence $\nu_1 = \nu_2$ by Theorem 1.6. So it follows that $\mu_1(E) = \mu_2(E)$ for every θ-Borel set E[3]. Hence $\mu_1 = \mu_2$.

Lemma 1.8. *If f is any integrable θ-Baire function, then there exists a sequence f_n of integrable simple θ-Baire functions such that $f_n \to f$ pointwise and $\int (f - f_n) dv \to 0$, where v denotes the given θ-Baire measure.*

Proof. If suffices to consider the case when $f \geq 0$. By definition of integrability, there exists a sequence f_n of integrable simple functions such that $0 \leq f_n \uparrow f$, and $\int f dv$ is defined to be sup $\int f_n dv$. Thus $\int f_n dv \uparrow \int f dv$; since $f - f_n \geq 0$ we have $\int (f - f_n) dv = \int f dv - \int f_n dv \to 0$

Theorem 1.9. *Let v be θ-Baire measure on X and suppose f is a v-integrable θ-Baire function. Then for every $\varepsilon > 0$, there exists a function g, given by $g = \sum_{j=1}^{m} a_j \chi_{A_j}$, where a_j's are real numbers and A_j's are mutually disjoint $H\text{-}G_\sigma^\theta$ sets such that $\int |f - g| dv \leq \varepsilon$.*

Proof. Without any loss of generality, by Lemma 1.8 we may assume $f = \sum_{1}^{m} \beta_j \chi_{B_j}$, where β_j's are real numbers and B_j's are mutually disjoint θ-Baire sets and $v(B_j) < +\infty$, for all j. If K is maximum of the finite set $\{|\beta_j| : 1 \leq j \leq m\}$ then clearly $|f| \leq K$.

Now, v being regular is inner regular, so for each B_j, there exists $H\text{-}G_\sigma^\theta$ set A_j such that $A_j \subset B_j$ and $v(B_j - A_j) \leq \frac{\varepsilon}{mK}$. Now, we define a function $g = \sum_{j=1}^{m} \beta_j \chi_{A_j}$.

Then
$$f - g = \sum_{j=1}^{m} \beta_j \left(\chi_{B_j} - \chi_{A_j}\right) = \sum_{j=1}^{m} \beta_j \chi_{B_j - A_j}$$

So
$$\int |f - g| \leq \sum_{j=1}^{m} |\beta_j| v\left(B_j - A_j\right) \leq \sum_{j=1}^{m} K \frac{\varepsilon}{mK} = \varepsilon$$

This Completes the theorem.

Our next theorem is an approximation of v-integrable function by a function in $\mathcal{H}$.

Theorem 1.10. *If f is any v-integrable function, where v is any θ-Baire measure on X, then for any $\varepsilon > 0$, there exists a function $g \in \mathcal{H}$ such that $\int |f - g| dv < \varepsilon$.*

Proof. In view of Theorem 1.9, we can assume $f = \sum_{j=1}^{m} \beta_j \chi_{A_j}$, where β_j's are real numbers and A_j's are mutually disjoint $H\text{-}G_\delta^\theta$ sets. Now corresponding to each χ_{A_j} (where A_j is an $H\text{-}G_\delta^\theta$ set) there exists a sequence $\{f_n^j\}$ of functions from $\mathcal{H}$ converging pointwise to χ_{A_j} [2]. Since linear combination of finite number of functions in $\mathcal{H}$ is again in $\mathcal{H}$, so finite linear combination of f_n^j's for j = 1, 2, ..., m is a sequence of functions say $\{f_n\}$ in $\mathcal{H}$ converging pointwise to f. Hence $\left\{\int f_n dv\right\}$ converges pointwise to $\int f dv$. So, for each $\varepsilon > 0$, there exists a continuous function with H-set support from $\{f_n\}$ say, $f_k = g$ such that $\int |f - g|\, dv < \varepsilon$. This completes the proof.

2. APPROXIMATION OF μ-INTEGRABLE FUNCTION BY A FUNCTION IN $\mathcal{H}$

We had studied extensively θ-Borel and θ-Baire functions along with θ-Borel and θ-Baire measures in [2] and [3]. The set of θ-Borel functions obviously contains the set of θ-Baire functions which is why it becomes quite pertinent to ask whether θ-Borel functions can be approximated by θ-Baire functions. The answer is not only in the affirmative as can be seen from the subsequent discussion, but also, even more, every μ-integrable function where μ is a regular θ-Borel measure, can be approximated by a function in $\mathcal{H}$.

Proposition 2.1. *Let μ be a regular θ-Borel measure and E is any θ-Borel set, then there exists a θ-Baire set F such that $\chi_E = \chi_F$ a.e. $[\mu]$.*

Proof. It is sufficient to prove that for each θ-Borel set E, there exists a θ-Baire set F such that $\mu(E \,\Delta\, F) = 0$.

Case I : Let $\mu(E) < +\infty$. By regularity, there exists a sequence of H-sets $\{C_n\}$ such that $C_n \subset E$ and $\mu(C_n) \uparrow \mu(E)$. Let $G = \bigcup_{n=1}^{\infty} C_n$ then clearly $G \subset E$ and $\mu(G) = \mu(E)$ i.e., $\mu(E \setminus G) = 0$. Using regularity of μ, for each n; we may choose an $H\text{-}G_\delta^\theta$ set D_n such that $C_n \subset D_n$ and $\mu(D_n - C_n) = 0$ [3]; if we take $F = \bigcup_{n=1}^{\infty} D_n$, then F is a θ-Baire set such that $G \subset F$ and $\mu(F - G) = 0$; so $\mu(E \,\Delta\, F) = 0$.

Case II. $\mu(E) = \infty$; then $E \subset \bigcup_{i=1}^{\infty} C_i$ where C_i's are H-sets ; since $\mu(C_i) < +\infty$ for each i, μ is σ-finite ; thus there exists a sequence of θ-Borel sets $\{E_i\}$ of finite measures such that $E = \bigcup_{i=1}^{\infty} E_i$. Then by case I; for each n; there exists a θ-Baire set F_n such that $\mu(E_n \Delta F_n) = 0$; then if we take $F = \bigcup_{n=1}^{\infty} F_n$ clearly $\mu(E \Delta F) = 0$, since $E \Delta F \subset \bigcup_{n=1}^{\infty} E_n \Delta F_n$; obviously, F is a θ-Baire set.

Theorem 2.2. *If f is any θ-Borel function and μ is a regular θ-Borel measure on X, then there exists a θ-Baire function h such that $f = h$ a.e.*

Proof. Here f is any θ-Borel function, so there exists a sequence $\{f_n\}$ of simple θ-Borel functions converging pointwise to f. Hence by Proposition 2.1, there exists a sequence of simple θ-Baire functions $\{g_n\}$ such that $g_n = f_n$ a.e. $[\mu]$.

So $g_n \to f$ a.e. $[\mu]$. Let E be the set such that $\mu(E) = 0$ and $g_n \to f$ pointwise on $X \setminus E$.

Let us construct the set F_{mn}, by

$$F_{mn} = \bigcup_{i,j \geq n} \left\{ x : |g_i(x) - g_j(x)| \geq \frac{1}{m} \right\};$$

then clearly F_{mn} is a θ-Baire set for each m, n [2]. Consequently,

$$F = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} F_{mn}$$

is a θ-Baire set. Now, since g_n is convergent on $X \setminus E$, so $F \subset E$ and so $\mu(F) = 0$.

If we define $h_n = \chi_{X \setminus F} g_n$; then h_n is a θ-Baire function [2] and $h_n = g_n$ a.e. $[\mu]$ and hence $h_n \to f$ a.e. $[\mu]$. Then h, the pointwise limit of $\{h_n\}$ is a θ-Baire function such that $f = h$ a.e. $[\mu]$.

Finally, we prove that, each μ-integrable θ-Borel function can be approximated by functions in $\mathcal{H}$, i.e., corresponding to each μ-integrable θ-Borel function f, there exists a sequence $\{f_n\}$ of functions in $\mathcal{H}$ converging pointwise to f and in this case $\int f\, d\mu = \lim_n \int f_n d\mu$.

Theorem 2.3. *Let μ be a regular θ-Borel measure X and suppose f is a μ-integrable θ-Borel function, then for every $\varepsilon > 0$, there exists a function $h \in \mathcal{H}$ such that $\int |f - h|\, d\mu \leq \varepsilon$.*

Proof. By Theorem 2.2, there exists a θ-Baire function g such that $g = f$ a.e. $[\mu]$, then g is μ-integrable (by 0.4). Let ν be the θ-Baire restriction of μ, then g is ν-integrable [by Theorem 1.1], then for any $\varepsilon > 0$, there exists a function $h \in \mathcal{H}$ such that

$$\int |g-h|\, d\nu \le \varepsilon \qquad \text{[by 1.10]}$$

$$\Rightarrow \int |g-h|\, d\mu \le \varepsilon \qquad \text{[by 1.1]}$$

$$\Rightarrow \int |f-h|\, d\mu \le \varepsilon \qquad (\text{since } |f-h| \le |g-h| \text{ a.e. } [\mu])$$

This completes the theorem.

3. GENERALIZED REPRESENTATION THEOREM

Notation 3.1. If f is any real valued function on X and $E \subset X$, then we write $f \geq E$ if $f(x) \geq 1$ for all $x \in E$ and $f(x) \geq 0$ if $x \in X \setminus E$.

Theorem 3.2. *If F is a positive linear form on $\mathcal{H}$, then there exists a unique regular θ-Borel measure μ such that $F(f) = \int f\, d\mu$, for all $f \in \mathcal{H}$*

Proof. We define a set function

$$\lambda(H) = \inf\{F(f) : f \geq H \text{ and } f \in H\} \text{ for every } H\text{-set } H.$$

Since F is positive, $\lambda(H) \geq 0$ for every H-set H. Let H be any H-set and V be any θ-bounded θ-open set containing H. Then there exists a function $f \in \mathcal{H}$ such that $f(x) = 1$ for all $x \in H$ and $f(X \setminus V) = 0$ and $0 \leq f \leq 1$ (by 0.1). So $\lambda(H) \leq F(f) < \infty$. Therefore λ is finite.

To prove λ is monotone, let H_1 and H_2 are H-sets such that $H_1 \subset H_2$ and $f \geq H_2$, where $f \in \mathcal{H}$ (existence of such function is ensured by 0.1), then $f \geq H_1$ also and hence $F(f) \geq \lambda(H_1)$. Therefore $\inf\{F(f) : f \geq H_2\} \geq \lambda(H_1)$, i.e. $\lambda(H_2) \geq \lambda(H_1)$.

To prove λ is subadditive, let H_1 and H_2 be two H-sets, then if $f_1, f_2 \in \mathcal{H}$ be such that $f_1 \geq H_1$ and $f_2 \geq H_2$ (existence of such function is ensured by 0.1). Then $f_1 + f_2 \geq H_1 \cup H_2$.Clearly $f_1 + f_2 \in \mathcal{H}$, so $F(f_1) + F(f_2) = F(f_1 + f_2) \geq \lambda(H_1 \cup H_2)$.

Then clearly

$$\lambda(H_1) + \lambda(H_2) = \inf F(f_1) + \inf F(f_2) \geq \lambda(H_1 \cup H_2).$$

Using subadditivity, we shall prove that λ is additive. Suppose H_1 and H_2 are disjoint H-sets, then as X is θ-T_2, there exist θ-bounded θ-open sets $W_i \supset H_i$ such that $W_1 \cap W_2 = \phi$. So there exists $f_i \in \mathcal{H}$, for $i = 1, 2$ such that $f_i(H_i) = 1$ and $f_i(X \setminus W_i) = 0$ and $0 \leq f_i \leq 1$ (By 0.1). If $g \in \mathcal{H}$ be such that $g \geq H_1 \cup H_2$, then

$$\sum_{i=1}^{2} \lambda(H_i) \leq \sum_{i=1}^{2} F(gf_i) = F\left(g \sum_{i=1}^{2} f_i\right) \leq F(g)$$

and hence

$$\lambda(H_1) + \lambda(H_2) \leq \inf\{F(g) : g \geq H_1 \cup H_2\}$$

i.e., $$\lambda(H_1) + \lambda(H_2) \leq \lambda(H_1 \cup H_2).$$

Using subadditivity, additivity of λ follows. Therefore λ is a θ-content.

Next we shall show that λ is a regular θ-content. To prove this, let H be an H-set and $\varepsilon > 0$, then by definition of λ there exists a function $f \in \mathcal{H}$ such taht $f \geq H$ and

$$F(f) \leq \lambda(H) + \frac{\varepsilon}{2}.$$

If α is a real number, $0 < \alpha < 1$ and $K = f^{-1}[\alpha,\infty)$, then

$$H \subset f^{-1}[1,\infty) \subset f^{-1}(\alpha,\infty) \subset f^{-1}[\alpha,\infty) = K.$$

So K is an H-set [2] and since f is real valued continuous function, hence θ-continuous ; so $f^{-1}(\alpha,\infty)$ is θ-open [since (α, ∞) is θ-open]. Clearly, $\frac{1}{\alpha}f \geq K$ and $\frac{1}{\alpha}f \in \mathcal{H}$ also $\frac{1}{\alpha}f(x) \geq 1$ for all $x \in K$ and $\frac{1}{\alpha}f(x) = 0$ when $x \in X \setminus K$.

Now, $$\lambda(K) \leq F\left(\frac{1}{\alpha}f\right) = \frac{1}{\alpha}F(f) \leq \frac{1}{\alpha}\left\{\lambda(H) + \frac{\varepsilon}{2}\right\}.$$

By choosing α near enough to 1 we can write $\frac{1}{\alpha}\left(\lambda(H) + \frac{\varepsilon}{2}\right) \leq \lambda(H) + \varepsilon$. Therefore $\lambda(K) \leq \lambda(H) + \varepsilon$. Hence by definition of regular θ-content, λ is a regular θ-content. So there exists a unique regular θ-Borel measure μ such that $\mu(H) = \lambda(H)$ for every H-set H.

Next we show that if $f \in \mathcal{H}$ and $f \geq 0$, then $\int f\, d\mu \leq F(f)$ ...... (1). To prove this inequality, by linearity it is sufficient to prove for functions f such that $0 \leq f \leq 1$.

For a fixed positive integer m, and for any x for which $f(x) \in \left[\frac{i-1}{m}, \frac{i}{m}\right]$ we define

$$f_k(x) = \begin{cases} 1 \text{ if } 1 \leq k \leq i-1 \\ 0 \text{ if } i+1 \leq k \leq m \end{cases}$$

and $$f_i(x) = mf - (i-1).$$

Clearly, $f(x) = \frac{1}{m}\sum_{k=1}^{m} f_k(x)$, for every $x \in X$. Since f is continuous and hence θ-continuous, so the sets $U_k = f^{-1}\left(\frac{k}{m},\infty\right)$ are θ-open, for $k = 0, 1, 2, \ldots, m$ and also U_k's are monotone decreasing subsets of X with $U_m = \phi$ (as $0 \le f \le 1$).

We claim that $\mu(U_k) \le F(f_k)$. If H be an H-set contained in U_k, then $F_k \ge H$ and so

$$\mu(H) = \lambda(H) \le F(f_k);$$

and hence

$$\mu(U_k) = \sup \mu(H) \le F(f_k).$$

Therefore we can write

$$\begin{aligned}
F(f) &= F\left(\frac{1}{m}\sum_{k=1}^{m} f_k\right) \\
&= \frac{1}{m}\sum_{k=1}^{m} F(f_k) \\
&\ge \frac{1}{m}\sum_{k=1}^{m} \mu(U_k) \\
&= \sum_{k=1}^{m}\left(\frac{k}{m}-\frac{k-1}{m}\right)\mu(U_k) \\
&= \sum_{k=1}^{m-1}\frac{k}{m}[\mu(U_k)-\mu(U_{k+1})] \\
&= \sum_{k=1}^{m-1}\frac{k+1}{m}\mu(U_k - U_{k+1}) - \frac{1}{m}\mu(U_1) \\
&\ge \sum_{k=1}^{m-1}\int_{U_k - U_{k-1}} f\,d\mu - \tfrac{1}{m}\mu(U_1)
\end{aligned}$$

.

$$= \int_{U_1} f\,d\mu - \tfrac{1}{m}\mu(U_1)$$

$$\geq \int f\,d\mu - \tfrac{1}{m}\mu(U_0)$$

Since $\mu(U_0)$ is finite and m is arbitrary, we have $\int f\,d\mu \leq F(f)$.

To complete our proof, it remains to show that $F(f) \leq \int f\,d\mu$, for every $f \in \mathcal{H}$, because the reverse inequality i.e., $\int f\,d\mu \leq F(f)$ follows by applying that inequality to $-f$.

So let f be a function in $\mathcal{H}$ and H be its H-set support. The definition of λ shows the existence of a function $h \in \mathcal{H}$ such that $h \geq H$ and $F(h) \leq \lambda(H) + \varepsilon$. If $h_0 = \min\{h,1\}$, then

$$F(h_0) \leq F(h) \leq \lambda(H) + \varepsilon \leq \int h_0\,d\mu + \varepsilon$$

Since $h_0 \geq H$, then $fh_0 = f$. Since f is real valued continuous function and hence θ-continuous so f carries H-set to H-sets and hence is closed and bounded there, $|f(x)| \leq \beta$, for all $x \in X$, then $h_0\,(f + \beta) \geq 0$ and $h_0(f + \beta) \in \mathcal{H}$. So by (1),

$$\begin{aligned} \int (f+\beta)h_0\,d\mu &\leq F((f+\beta)h_0) \\ &= F(fh_0) + F(\beta h_0) \\ &= F(f) + \beta F(h_0) \end{aligned}$$

Therefore,

$$\begin{aligned} \int (f+\beta)h_0\,d\mu &= \int f\,d\mu + \beta \int h_0\,d\mu \\ &\leq F(f) + \beta F(h_0) \end{aligned}$$

i.e.,

$$\begin{aligned} F(f) &\geq \int f\,d\mu + \beta\left[\int h_0\,d\mu - F(h_0)\right] \\ &\geq \int f\,d\mu - \beta\varepsilon \quad [\text{since, } F(h_0) \leq \int h_0\,d\mu + \varepsilon] \end{aligned}$$

Since ε is arbitrary,

$$\int f\,d\mu \leq F(f)$$

Hence the theorem is complete.

ACKNOWLEDGEMENT

The author expresses his earnest gratitude to Prof. S. Ganguly, Department of Pure Mathematics, University of Calcutta, for his guidance throughout the preparation of this paper and to Dr. C. K. Basu, Department of Mathematics, Kalyani University, for his valuable suggestions.

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