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ON SOME CONGRUENCES IN SEMIRING

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ABSTRACT : The concept of p-ideals in a semiring R was first introduced by Mukhopadhyay and Ghosh [4,5] and subsequently various characteristic features of a p-ideal in different classes of semirings were obtained in [4], [5], [6]. In the present paper, a congruence ρ_I induced by an ideal I of R is provided and certain basic results corresponding to this concept including the nature of the congruence when I is a p-ideal in particular, are established. In the last section, yet another congruence, Bourne p-congruence, compatible with the concept of p-ideals in a semiring is established and studied at length.

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1. INTRODUCTION AND PRELIMINARIES

A semiring is a non-empty set S together with two binary operations, called "addition", +, and "multiplication" (usually denoted by juxtaposition), such that S is multiplicatively a semigroup and additively a commutative semigroup and the multiplication is distributed accross the addition both from the left and from the right. a semiring is said to be *commutative* if it is multiplicatively commutative. An additively cancellative semiring is called a *halfring.* An *inversive semiring* [7] S is a semiring in which $(S, +)$ is an inverse semigroup, i.e., for each $a \in S$, there is a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. It is well known [3] that, in an inversive semiring S, we have $(ab)' = a'b = ab'$ and $(a + b)' = a' + b'$. $E^+(S)$ denotes the set of all additive idempotents of S. If an ideal of the semiring contains $E^+(S)$ then it is called a *full* ideal. A semiring S ia said to have an identity, if there exists $1_s \in S$ such that, $1_s x = x 1_s = x$ for each $x \in S$. The zero element of S, denoted by 0, is called an *absorbing zero* if $a0 = 0a = 0$ for all $a \in S$. A *k-ideal* [2] I of a semiring S is an ideal of S such that, if $a \in I$, $x \in S$ and $a + x \in I$, then $x \in I$. A *congruence* ρ on S is an equivalence relation on S such that $a \rho b$ in S implies $(a + c) \rho (b + c)$, $ac \rho bc$ and $ca \rho cb$ hold for all $a, b, c \in S$. The *congruence class* of an element $a \in S$ under ρ is denoted by $a\rho$. The *quotient semiring* $\{a\rho : a \in S\}$ of S under ρ is defined usually and is denoted by S/ρ. It is interesting to point out that, unlike rings, ideals and congruences, in general, do not correspond bijectively in semirings. Throughout this paper $\mathbb{N}$ represents the set of natural numbers.

Mukhopadhyay and Ghosh [4] introduced the idea of p-ideals in semirings. We recall the following definitions and results that will be useful in the sequel.

Definition 1.1. [4,5] An ideal I of a semiring S is called a *p-ideal* if for some $x \in S$, $n \in \mathbb{N}$,

$$nx + a = (n + 1)x \text{ and } a \in I \text{ implies } x \in I.$$

In particular, if S is *inversive,* then the definition boils down to the following:

$$\text{if for some } x \in S,\ a + x = 2x,\ a \in I \text{ then } x \in I.$$

Observe that all p-ideals are not k-ideals, as the ideal $I = 3Z_0^+ \setminus \{3\}$ is not a k-ideal, in the halfring Z_0^+ of all positive integers with zero but it is a p-ideal, as all ideals of a halfring are p-ideals. We also note that k-ideals are not p-ideals in general. Indeed, in the semiring $(Z^+, \max, \min)$, $I_n = \{1, 2, 3, ..., n\}$ is a k-ideal for any $n \in Z^+$ but not a p-ideal. It is interesting to see that in an inversive semiring S, $E^+(S)$ is a p-ideal. In fact, any full ideal of an inversive semiring is a p-ideal.

Following result was proved in [4, 5].

Proposition 1.2. *In an inversive semiring S an ideal I is a p-ideal if and only if* $I = I + E^+(S)$.

In search of the smallest p-ideal containing a given ideal of a semiring the following definition was given in [4].

Definition 1.3. For any subsemiring R of a semiring S, we define,

$$\hat{R} = \{x \in S \mid a + nx = (n + 1)x, \text{ for some } n \in \mathbb{N},\ a \in R\}.$$

Proposition 1.4. [4] *For any two ideals I, J of a semiring S we see that,* $\hat{I}$ *is a p-ideal of S such that* $\hat{\hat{I}} = \hat{I}$; $I \subseteq \hat{I}$ *if* $I \subseteq J$ *then* $\hat{I} \subseteq \hat{J}$; *indeed,* $\hat{I}$ *is the smallest p-ideal of S containing I.*

Note that in case of an *inversive semiring S,* for an ideal I of S,

$$\hat{I} = \{x \in S \mid a + x = 2x \text{ for some } a \in I\}.$$

This leads us to the following result immediately:

Corollary 1.5. [4, 5] *In an inversive semiring S, we have* $\hat{I} = I + E^+(S)$, *for any ideal I.*

2. CONGRUENCE INDUCED BY A p-IDEAL

In keeping with the spirit of the concept of p-ideals in a semiring, let us now introduce the concept of a new kind of congruence induced by an ideal in a semiring S with absorbing zero.

Definition 2.1. Let I be an ideal in a semiring S. We define a *relation* ρ_I on S, *induced by* I, as follows: for $a, b \in S$,

$$a\rho_I b \Leftrightarrow na + x + b = (n+1)a$$
$$mb + y + a = (m+1)b,$$

for some $x, y \in I$ and $m, n \in \mathbb{N}$.

Proposition 2.2. ρ_I *(as defined above) is a congruence on* S.

Proof. We first show that ρ_I is an equivalence relation on S. Since S has an absorbing zero, it follows that ρ_I is reflexive; whereas symmetry of ρ_I is obvious from its definition. Towards transitivity, let $a\,\rho_I\,b$, $b\,\rho_I\,c$ for some $a, b, c \in S$. From the definition of ρ_I we have,

(1) $na + x + b = (n+1)a$, and

(2) $mb + y + a = (m+1)b$, for some $x, y \in I$; $m, n \in \mathbb{N}$ and

(3) $rb + z + c = (r+1)b$, and

(4) $sc + w + b = (s+1)c$, for some $z, w \in I$; $r, s \in \mathbb{N}$.

Now, from (1) we have,

$$(1+r)na + (1+r)x + (1+r)b = (1+r)(n+1)a$$

i.e. $(1+r)na + (1+r)x + rb + z + c = (1+r)(n+1)a$ [by (3)]

i.e. $rna + na + x + rx + rb + z + c = (1+r)(n+1)a$

i.e. $r(na + x + b) + na + x + z + c = (nr + n + r + 1)a$

i.e. $r(n+1)a + na + x + z + c = (nr + n + r + 1)a$ [by (1)]

i.e. $(nr + r + n)a + (x + z) + c = (nr + n + r + 1)a$

i.e. $ka + (x + z) + c = (k+1)a$,

where $x + z \in I$ and $k = nr + n + r \in \mathbb{N}$.

Again, we have, from (4),

$$(m+1)sc + (m+1)w + (m+1)b = (m+1)(s+1)c$$

i.e. $msc + sc + mw + w + mb + y + a = (ms + s + m + 1)c$ [by (2)]

i.e. $m(sc + w + b) + sc + w + y + a = (ms + s + m + 1)c$

i.e. $m(s + 1)c + sc + w + y + a = (ms + s + m + 1)c$ [by (4)]

i.e. $tc + (w + y) + a = (t + 1)c$,

where $w + y \in I$ and $t = ms + s + m \in \mathbb{N}$.

Consequently, we have $a \rho_I c$ i.e., ρ_I is a transitive relation and hence an equivalence relation on S. We now show that, actually ρ_I is a congruence relation on S. Let, for some $a, b, c, d \in S$, $a \rho_I b$ and $c \rho_I d$ hold. Then we have,

(5) $na + x + b = (n + 1)a$, and

(6) $mb + y + a = (m + 1)b$, for some $x, y \in I$; $m, n \in \mathbb{N}$ and

(7) $sc + w + d = (s + 1)c$, and

(8) $td + z + c = (t + 1)d$, for some $w, z \in I$; $s, t \in \mathbb{N}$.

It is easy to see that,

$$k(a + c) + (x + w) + (b + d) = (k + 1)(a + c),$$

where $x + w \in I$ and $k = \max(n, s) \in \mathbb{N}$, and also,

$$r(b + d) + (z + y) + (a + c) = (r + 1)(b + d),$$

where $y + z \in I$ and $r = \max(m, t) \in \mathbb{N}$, which together imply that, $(a + c)\rho_I(b + d)$ hold.

We further see that, for some $c \in S$, we have from (5) and (6) that,

$$nac + xc + bc = (n + 1)ac \text{ and}$$

$$mbc + yc + ac = (m + 1)bc,$$

where $xc, yc \in I$ as I is an ideal, showing that, $(ac)\rho_I(bc)$ holds. In a similar manner, it can be also shown that, $(ca)\rho_I(cb)$ holds, whence ρ_I is a congruence on S.

Remark. It is interesting to note that, when I is a p-ideal, elements of I are ρ_I related only to elements of I i.e., $b \in I$ with $a \rho_I b$ implies that $a \in I$. Indeed, from $na + x + b = (n + 1)a$, for some $x \in I$ and $n \in \mathbb{N}$ [as in (5)], with $b \in I$ we have, $(x + b) \in I$, whence, as I is a p-ideal we get, $a \in I$. So we see that ρ_I saturates the p-ideal I. However, we point out that, for any two elements $a, b \in I$, it is not mandetory that $a \rho_I b$ holds;

i.e. a p-ideal I is being actually partitioned into disjoint ρ_I classes. Indeed, we have the following:

Proposition 2.3. *I constitutes a ρ_I class if and only if I is a subring of S.*

Proof. Let I be a subring of S, and let $a, b \in I$ be any two elements of I. Then choosing $x = a - b \in I$ and $y = b - a \in I$ it can be easily seen that the conditions (5) and (6) of Proposition 2.2 are automatically satisfied, for any $m, n \in \mathbb{N}$ whence $a\rho_I b$ holds, showing thereby that, I is a single ρ_I class.

Conversely, let I be a ρ_I class, i.e., for all $a, b \in I$, $a\rho_I b$ holds. Since I is an ideal of S, we have $0 \in I$, which indicates, $a\rho_I 0$ must hold good; whence for any $a \in I$ we get that there exists some $y \in I$ such that $a + y = 0$, showing that I is a subring of S.

Proposition 2.4. For any ideal I of a semiring S, $S/I = S/\hat{I}$.

Proof. Since $I \subseteq \hat{I}$, $a\rho_I b \Rightarrow a\rho_{\hat{I}} b$ for any $a, b \in S$.

Conversely, let $a\rho_{\hat{I}} b$ hold for some $a, b \in S$. Then,

(1) $na + x + b = (n + 1)a$ and

(2) $mb + y + a = (m + 1)b$

for some $x, y \in \hat{I}$ and $m, n \in \mathbb{N}$. Now, since x, y $\in \hat{I}$, we have that

(3) $kx + i_1 = (k + 1)x$ and

(4) $ly + i_2 = (l + 1)y$

for some $i_1 + i_2 \in I$ and $k, l \in \mathbb{N}$. Then from (3), we have that,

$$kx + i_1 + (k + 1)b + n(k + 1)a = (k + 1)x + (k + 1)b + n(k + 1)a$$

i.e. $k(na + x + b) + na + i_1 + b = (k + 1)(na + x + b)$

i.e. $k(n + 1)a + na + i_1 + b = (k + 1)(n + 1)a$ [from (1)]

i.e. $pa + i_1 + b = (p + 1)a$ ...(1)

where $p = kn + k + n \in \mathbb{N}$.

Now from (4), we have that,

$$ly + i_2 + (l + 1)a + m(l + 1)b = (l + 1)y + (l + 1)a + m(l + 1)b$$

i.e., $l(mb + y + a) + mb + i_2 + a = (l + 1)(mb + y + a)$

i.e., $l(m+1)b + mb + i_2 + a = (l+1)(m+1)b$ [from (2)]

i.e., $qb + i_2 + a = (q+1)b$ ...(*ii*)

where $q = lm + l + m \in \mathbb{N}$.

Hence from (i) and (ii), we conclude that $a\rho_I b$ holds.

Proposition 2.5. For any ideal I of a semiring S and for any $a\rho_I \in P^+(S/I)$ there exists $i \in P^+(S)$ such that $b = a + i \in P^+(S)$ and $a\rho_I = b\rho_I$.

Proof. Let $a\rho_I \in P^+(S/I)$. Then, for some $n \in \mathbb{N}$, $na\rho_I = (n+1)a\rho_I$. Hence we have the following:

$$k(na) + i + (n+1)a = (k+1)na, \quad l(n+1)a + j + na = (l+1)(n+1)a$$

for some $k, l \in \mathbb{N}$ and $i, j \in I$. Suppose, $p = \max.\{l, k\}$. Then,

(1) $p(na) + i + (n+1)a = (p+1)na$ and

(2) $p(n+1)a + j + na = (p+1)(n+1)a$

Now from (1), we have that,

$$(pn+n)a + (a+i) = (pn+n)a \text{ ...(iii)}$$

i.e., $(pn+n)a + (a+i) + (pn+n)i = (pn+n)a + (pn+n)i$

i.e., $(pn+n)(a+i) + (a+i) = (pn+n)(a+i)$

i.e., $(pn+n+1)(a+i) = (pn+n)(a+i)$

i.e. $b = a + i \in P^+(S)$.

From (2), we have that,

$$(pn+n+p)a + j = (pn+n+p+1)a$$

i.e., $(pn+n+p)a + j + (a+i) = (pn+n+p+1)a$ [from (iii)]

i.e., $ma + j + b = (m+1)a$ ...(α)

where $m = pn + n + p \in \mathbb{N}$.

Moreover, $(a+i) + i + a = 2(a+i)$, i.e., $b + i + a = 2b$ ...(β)

(α) and (β) together imply that $a\rho_I = b\rho_I$.

Remark 2.6. Since for any $a \in P^+(S)$, $a\rho_I$ $P^+(S/I)$, so from Proposition 2.5., $P^+(S/I) = \{a\rho_I : a \in P^+(S)\}$.

Proposition 2.7. For ay ideal I of a semiring S, $P = \bigcup_{a \in P^+(S)} a\rho_I$ is a p-ideal of S, contained in $\hat{I}$.

Proof. For any, $p, q \in P$, $\exists a, b\ P^+(S)$ such that $p \in a\rho_I$ and $q \in b\rho_I$. Then, $p + q \in (a + b)\rho_I$ and $xpy \in (xay)\rho_I$ for any $x, y \in S$ and hence P is an ideal of S.

Let $x \in \hat{P}$ be arbitrary. Then, $\exists b \in P$ and $m \in \mathbb{N}$ such that $mx + b = (m + 1)x$. Hence, $mx\rho_I + b\rho_I = (m + 1)x\rho_I$ whence $x\rho_I \in P^+(S/I)$ (since $P^+(S/I)$ is a p-ideal of the semiring S/I and $b\rho_I \in P^+(S/I)$). Hence $x \in P$. So, P is a p-ideal of S. Now $b \in P = \bigcup_{a \in P^+(S)} a\rho_I \Rightarrow$ $\exists a \in P^+(S)$ such that $b \in a\rho_I$. Thus, there exist $k \in \mathbb{N}$ and $j \in I$ such that $kb + j + a = (k + 1)b$. Thus $b \in P$ (since P is p-ideal of S and $P^+(S) \subseteq I$).

Proposition 2.8. *In an inversive semiring S, the ρ_I classes of a full k-ideal I, and hence ρ_I classes of a p-ideal I which is also a k-ideal, are precisely the idempotents of S/ρ_I.*

Proof. We recall that in an inversive semiring S, any full ideal is a p-ideal Let I be a full k-ideal in S. We show that,

$$E^+(S/\rho_I) = I/\rho_I.$$

Indeed, let $a\rho_I \in E^+(S/\rho_I)$, for some $a \in S$, so that, $a\rho_I 2a$ gives $3a + x = 2a$ and $3a + y = 4a$ for some $x, y \in I$ where the second relation gives $a + y = 2a$ on addition of adequate number of a' in both sides, giving that, $a \in I$ i.e., $a\rho_I \in I/\rho_I$ so that,

$$E^+(S/\rho_I) \subseteq I/\rho_I.$$

Conversely, we assume $a\rho_I \in I/\rho_I$ so that, $a \in I$ which is a full k-ideal of S. Then $a\rho_I 2a$ holds; indeed, I being a full k-ideal we have $a' \in I$ so that, $a + a' + 2a = 2a$ and $2a + a + a = 4a$ establishes our claim; hence $a\rho_I \in E^+(S/\rho_I)$ i.e.,

$$I/\rho_I \subseteq E^+(S/\rho_1).$$

Hence the result follows.

3. BOURNE p-CONGRUENCE

Definition 3.1. Let I be an ideal in a semiring S. We define a *relation* ρ_I on S, *induced by* I, as follows: for $a, b \in S$,

$$a\rho_I b \Leftrightarrow na + x_1 + b = (n+1)a + x_2$$

$$mb + y_1 + a = (m+1)b + y_2,$$

for some $x_1, x_2, y_1, y_2 \in I$ and $m, n \in \mathbb{N}$.

Remark 3.2. For any ideal I of a semiring S, if $\equiv_I$ denotes the Bourne relation [1] on S, then $\equiv_I \subseteq \rho_I$. Indeed, let for any $a, b \in S$, $a \equiv_I b$. Then, $\exists i, j \in I$ such that $a + i = b + j$ and so by adding a and b separately to both the sides, we get, $a + j + b = 2a + i$ and $b + i + a = 2b + j$ whence $a\rho_I b$ holds.

Proposition 3.3. *ρ_I (as defined above) is a congruence on S.*

Proof. We first show that ρ_I is an equivalence relation on S. Since S has an absorbing zero, it follows that, ρ_I is reflexive; whereas symmetry of ρ_I is obvious from its definition. Towards transitivity, let $a\,\rho_I b$, $b\,\rho_I c$ for some $a, b, c \in S$. From the definition of ρ_I we have,

(1) $na + x_1 + b = (n+1)a + x_2,$ and

(2) $mb + y_1 + a = (m+1)b + y_2,$ for some $x_1, x_2, y_1, y_2 \in$ I; $m, n \in \mathbb{N}$ and

(3) $rb + z_1 + c = (r+1)b + z_2,$ and

(4) $sc + w_1 + b = (s+1)c + w_2,$ for some $z_1, z_2, w_1, w_2 \in I$; $r, s \in \mathbb{N}$.

Now from (1) we have that,

$$(r+1)na + (r+1)x_1 + (r+1)b + z_2 = (r+1)(n+1)a + (r+1)x_2 + z_2$$

i.e., $(r+1)na + (r+1)x_1 + rb + z_1 + c = (r+1)(n+1)a + (r+1)x_2 + z_2$ [from 3]

i.e., $na + r(na + x_1 + b) + z_1 + x_1 + c = (r+1)(n+1)a + (r+1)x_2 + z_2$

i.e., $na + r[(n+1)a + x_2] + x_1 + z_1 + c = (r+1)(n+1)a + (r+1)x_2 + z_2$ [from 1]

i.e., $(rn + n + r)a + [rx_2 + x_1 + z_1] + c = (rn + n + r + 1)a + [(r+1)x_2 + z_2]$

i.e., $pa + i_1 + c = (p+1)a + i_2$ ... (α)

where $p = rn + n + r \in \mathbb{N}$ and $i_1 = rx_2 + x_1 + z_1$, $i_2 = (r+1)x_2 + z_2 \in I$.

Again, from (4) we have that,

$$(m+1)sc + (m+1)w_1 + (m+1)b + y_2 = (m+1)(s+1)c + (m+1)w_2 + y_2$$

i.e., $(m+1)sc + (m+1)w_1 + mb + y_1 + a = (m+1)(s+1)c + (m+1)w_2 + y_2$ [from 2]

i.e., $(m + 1)sc + (m + 1)w_1 + mb + y_1 + a = (m + 1)(s + 1)c + (m + 1)w_2 + y_2$

i.e., $sc + m(sc + w_1 + b) + w_1 + y_1 + a = (m + 1)(s + 1)c + (m + 1)w_2 + y_2$

i.e., $sc + m[(s + 1)c + w_2] + w_1 + y_1 + a = (m + 1)(s + 1)c + (m + 1)w_2 + y_2$ [from 4]

i.e., $(ms + m + s)c + [mw_2 + w_1 + y_1] + a = (ms + m + s + 1)c + [(m + 1)w_2 + y_2]$

i.e., $qc + j_1 + a = (q + 1)c + i_2$ ... (β)

where $q = ms + m + s \in \mathbb{N}$ and $j_1 = mw_2 + w_1 + y_1$, $i_2 = (m + 1)w_2 + y_2 \in I$.

Clearly (α) and (β) together imply that $a\rho_I c$ holds i.e., the relation ρ_I is transitive. So, ρ_I is an equivalence relation. We now show that, actually ρ_I is a congruence relation of S. Let, for some $a, b, c, d \in S$, $a\rho_I b$ and $c\rho_I d$ hold. Then we have,

(5) $na + x_1 + b = (n + 1)a + x_2$, and

(6) $mb + y_1 + a = (m + 1)b + y_2$, for some $x_1, x_2, y_1, y_2 \in I$; $m, n \in \mathbb{N}$ and

(7) $sc + w_1 + d = (s + 1)c + w_2$, and

(8) $td + z_1 + c = (t + 1)d + z_2$, for some $w_1, w_2, z_1, z_2 \in I$; $s, t \in \mathbb{N}$.

It is easy to see that,

$$k(a + c) + (x_1 + w_1) + (b + d) = (k + 1)(a + c) + (x_2 + w_2),$$

where $x_1 + w_1, x_2 + w_2 \in I$ and $k = \max(n, s) \in \mathbb{N}$, and also,

$$r(b + d) + (y_1 + z_1) + (a + c) = (r + 1)(b + d) + (y_2 + z_2),$$

where $y_1 + z_1, y_2 + z_2 \in I$ and $r = \max(m, t) \in \mathbb{N}$, which together imply that, $(a + c)\rho_I(b + d)$ hold.

we further see that, for some $c \in S$, we have from (5) and (6) that,

$$nac + x_1c + bc = (n + 1)ac + x_2c \quad \text{and}$$

$$mbc + y_1c + ac = (m + 1)bc + y_2c,$$

where $x_1c, x_2c, y_1c, y_2c \in I$ as I is an ideal, showing that, $(ac)\rho_I(bc)$ holds. In a similar maner, it can be also shown that, $(ca)\rho_I(cb)$ holds, whence ρ_I is a congruence on S.

Proposition 3.4. For an ideal I of a semiring S, $P = \{a \in S : a\rho_I \in P^+(S/I)\}$ is a p-ideal of S containing I. also, $\bar{\hat{I}} = \bar{P}$.

Proof. Let $a, b \in P$. Then $\exists n, m \in \mathbb{N}$ such that $na\rho_I = (n+1)a\rho_I$ and $mb\rho_I = (m+1)b\rho_I$ whence $k(a+b)\rho_I = (k+1)(a+b)\rho_I$ where $k = max.\{n, m\}$. thus $a + b \in P$. For any $x \in S$, $na\rho_I = (n+1)a\rho_I \Rightarrow nxa\rho_I = (n+1)xa\rho_I$, $nax\rho_I = (n+1)ax\rho_I \Rightarrow xa, ax \in P$. So, P is an ideal of S.

Let $x \in S$ such that $nx + a = (n+1)x$ for some $a \in P$ and $n \in \mathbb{N}$. Then, $nx\rho_I + a\rho_I = (n+1)x\rho_I \Rightarrow x\rho_I \in P^+(S/I)$ (since, $P^+(S/I)$ is a p-ideal of S/I and $a\rho_I \in P^+(S/I)$) $\Rightarrow x \in P$. So, P is a p-ideal of S.

Since, $0 \in I$, for any $i \in I$, $i + i + 0 = 2i + 0$ and $0 + i + i = 0 + 2i$ which together imply that, $i\rho_I 0$ holds for all $i \in I$. Hence, $I \subseteq 0\rho_I$. Clearly, $\bigcup_{a \in P^+(S)} a\rho_I \subseteq P$; so, in particular $0\rho_I \subseteq P$. Consequently, $I \subseteq P$. So, $\bar{\hat{I}} \subseteq \bar{\hat{P}} = \bar{P}$ (since, P is a p-ideal of S). Conversely, let $x \in \bar{P}$. Then, $x + p = q$ for some $p, q \in P$. Now, $p \in P \Rightarrow np\rho_I = (n+1)p\rho_I$ for some $n \in \mathbb{N} \Rightarrow k(n+1)p + i + np = (k+1)(n+1)p + j$ for some $i, j \in I$ and $k \in \mathbb{N} \Rightarrow$ $(kn + k + n)p + i + (kn + k + n)j = (kn + k + n + 1)p + j + (kn + k + n)j \Rightarrow$ $(kn + k + n)(p + j) + i = (kn + k + n + 1)(p + j) \Rightarrow p + j \in \hat{I} \Rightarrow p \in \hat{I}$. Similarly, $q \in \hat{I}$. Thus, $x + p = q \Rightarrow x \in \hat{I}$.

Proposition 3.5. If I is a p-ideal of S, then $S/I = S/\bar{I}$ with $P^+(S/I) = \{0\rho_1\}$.

Proof. Since, $I \subseteq \bar{I}$, it is immediate to write that $a\rho_I b \Rightarrow a\rho_{\bar{I}} b$ for any, $a, b \in S$. Conversely, let $a, b \in S$ such that $a\rho_{\bar{I}} b$ holds. Then,

(1) $na + x_1 + b = (n+1)a + x_2$,

(2) $mb + y_1 + a = (m+1)b + y_2$,

for some $x_1, x_2, y_1, y_2 \in \bar{I}$; m, n $\mathbb{N}$. Moreover,

(3) $x_1 \in \bar{I} \Rightarrow x_1 + i_1 = i_1'$,

(4) $x_2 \in \bar{I} \Rightarrow x_2 + i_2 = i_2'$,

(5) $y_1 \in \bar{I} \Rightarrow y_1 + i_1 = j_1'$,

(6) $y_2 \in \bar{I} \Rightarrow y_2 + i_2 = j_2'$,

for some $i_k, i_k', j_k, j_k' \in I(k = 1, 2)$; $m, n \in \mathbb{N}$. Now, from (1), (3), (4) we have that,

$na + (x_1 + i_1) + i_2 + b = (n + 1)a + (x_2 + i_2) + i_1 \Rightarrow na + (i_1' + i_2) + b = (n + 1)a + (i_2' + i_1)$

$\Rightarrow na + i + b = (n + 1)a + i'$ ... (α), where $i = i_1' + i_2$, $i' = i_2' + i_1 \in I$.

Again, from (2), (5), (6) we have that,

$mb + (y_1 + i_1) + i_2 + a = (m + 1)b + (y_2 + i_2) + i_1 \Rightarrow mb + (j_1' + j_2) + a =$

$(m + 1)b + (j_2' + j_1) \Rightarrow mb + j + a = (m + 1)b + j'$ ... (β), where $j = j_1' + i_2$, $j' =$

$j_2' + i_1 \in I$.

Thus, from (α) and (β), $a\rho_I b$ holds.

Now, from Proposition 3.3 $\bar{I} = \bar{P}$ (since, I is a p-ideal). Consequently, $\bar{I} \subseteq \{a \in S:$ $a\rho_{\bar{I}} \in P^+(S/\bar{I})\}$ (Proposition 3.) $= \{a \in S : a\rho_I \in P^+(S/I)\} = P \subseteq \bar{P} = \bar{I}$ whence $\bar{I}$ $= P$. It is easy to show that $0\rho_I$ is a k-ideal of S. Hence, $P = \bar{I} \subseteq 0\rho_I \subseteq P \Rightarrow P = 0\rho_I$ $\Rightarrow P^+(S/I) = \{0\rho_I\}$.

Proposition 3.6. If I is a maximal p-ideal of a commutative semiring S, such that $\bar{I} \neq S$, then S/I is a p-semifield.

Proof. Since, I is a maximal p-ideal and $\bar{I} \neq S$, so $I = \bar{I} = 0\rho_I$ and thus $P^+(S/I) = \{0\rho_I\}$ $= \{I\}$. Let J be a p-ideal of S/I such that $P^+(S/I) \subseteq J \subseteq S/I$. Let us write, $J_0 = \{a \in S :$ $a\rho_I \in J\}$. Clearly, J_0 is p-ideal of S. Moreover, since $\{I\} = P^+(S/I) \subseteq J$, so $I \subseteq J_0$. Then, by the maximality of I as p-ideal, $J_0 = S$. So, $J = S/I$. Hence, S/I is a p-semifield.

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