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ON PRECONTINUOUS AND α-PRECONTINUOUS MAPPINGS

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ABSTRACT : Precontinuity and α-precontinuity of mappings in topological spaces are considered. Another properties of these types of mappings and interrelationships with some other types are studied. Some observations concerning Hausdorffness, normality and β-connectedness of spaces are given.

Key words and phrases : Preopen, α-open, semi-open, simply open sets; α-precontinuity, precontinuity, semi-continuity, α-continuity, a.c.S.; submaximal, β-connected, Hausdorff, normal, $\mathcal{D}$-spaces.
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1. INTRODUCTION

Mashhour et al. [28] introduced the notion of precontinuous mapping which coincides with almost continuity in the sense of Husain [16] (briefly: a.c.H.). Quite recently,Beceren and Noiri [4] have defined α-precontinuous mappings which are of stronger form of continuity than those precontinuous. Some interrelationships with other known classes of mappings can be found in [4]. In present paper we continue investigations concerning α-precontinuous and precontinuous mappings in context of several types of 'open' and 'continuous' mappings.

2. PRELIMINARIES

Throughout the paper, by (X, τ), (Y, σ), ... we mean topological spaces (briefly: spaces) on which no separation axioms are assumed unless explicitly stated. The Cartesian product topology for spaces (X, τ) and (Y, σ) will be denoted by $\tau \times \sigma$. Let S be a subset of an (X, τ). The closure of S and the interior of S (both in (X, τ)) are denoted by $\mathrm{cl}_\tau(S)$ (or $\mathrm{cl}(S)$) and $\mathrm{int}_\tau(S)$ (or int (S)), respectively. A subset $S \subset X$ is said to be ***regular open*** (resp. ***regular closed***) in (X, τ) if $S = \mathrm{int}\,(\mathrm{cl}(S))$ (resp. $S = \mathrm{cl}(\mathrm{int}(S))$). The family of all closed (resp. regular open, regular closed) subsets of a space (X, τ) will be denoted by $\mathrm{c}(X, \tau)$ (resp. RO (X, τ), $\mathrm{RC}(X, \tau)$). A subset S of (X, τ) is said to be ***α-open*** [32] (resp. ***semi-open*** [21], ***preopen*** [28], ***semi-preopen*** [2] or equivalently ***β-open*** [1]) if $S \subset \mathrm{int}\,(\mathrm{cl}(\mathrm{int}(S)))$ (resp. $S \subset \mathrm{cl}(\mathrm{int}(S))$, $S \subset \mathrm{int}(\mathrm{cl}(S))$, $S \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(S)))$). The complement of an α-open (resp. semi-open, preopen, semi-preopen) set is called ***α-closed*** (resp. ***semi-closed, preclosed, semi-preclosed***). The family of all α-open (resp. semi-open, preopen, semi-preopen) subsets of (X, τ) will be denoted by

τ^α (resp. $SO(X, \tau)$, $PO(X, \tau)$, $SPO(X, \tau)$). The family of all semi-closed (resp. preclosed, semi-preclosed) subsets of (X, τ) we denote by $SC(X, \tau)$ (resp. $PC(X, \tau)$, $SPC(X, \tau)$). The following incusions are known for any space (X, τ) : $\tau \subset \tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$ ([39, Lemma 3.1] or [45, Lemma 2]), $SO(X, \tau) \subset SPO(X, \tau)$ and $PO(X, \tau) \subset SPO(X, \tau)$. For each (X, τ) the family τ^α forms a topology on X [32, Proposition 2] which is strictly distinct from τ, in general. We have $(\tau^\alpha)^\alpha = \tau^\alpha$ [32, Proposition 10] for each (X, τ). If $\mathcal{A} \subset SO(X, \tau)$ (resp. $\mathcal{A} \subset PO(X, \tau)$), then $\cup \mathcal{A} \in SO(X, \tau)$ [21, Theorem 2] (resp. $\cup \mathcal{A} \in PO(X, \tau)$ [28]).

With a standard method, for any space (X, τ) and a subset $S \subset X$ one defines [2] the ***preinterior*** of S ($\mathrm{pint}_\tau(S)$), the ***semi-preinterior*** of S ($\mathrm{spint}_\tau(S)$), the ***preclosure*** of S ($\mathrm{pcl}_\tau(S)$), and the ***semi-preclosure*** of S ($\mathrm{spcl}_\tau(S)$) as respectively: the union of all preopen (resp. semi-preopen) subsets of (X, τ) contained in S and the intersection of all preclosed (resp. semi-preclosed) subsets of (X, τ) containing S.

The following formulas are known: $\mathrm{pcl}_\tau(S) = S \cup \mathrm{cl}_\tau(\mathrm{int}_\tau(S))$ [2, Theorem 1.5(e)], $\mathrm{pint}_\tau(S) = S \cap \mathrm{int}_\tau(\mathrm{cl}_\tau(S))$ [2, Theorem 1.5(f)]. A mapping $f: (X, \tau) \to (Y, \sigma)$ is said to be ***α-precontinuous*** [4] (resp. ***precontinuous*** [28], ***semicontinuous*** [21], ***α-continuous*** [30], ***α-irresolute*** [27], ***almost continuous in the sense of Singal and Singal*** [48] or briefly ***a.c.S.***) if $f^{-1}(V) \in PO(X, \tau)$ (resp. $f^{-1}(V) \in PO(X, \tau)$, $f^{-1}(V) \in SO(X, \tau)$, $f^{-1}(V) \in \tau^\alpha$, $f^{-1}(V) \in \tau^\alpha$, $f^{-1}(V) \in \tau$) for every set $V \in \sigma^\alpha$ (resp. $V \in \sigma$, $V \in \sigma$, $V \in \sigma$, $V \in \sigma^\alpha$, $V \in RO(X, \tau)$ [48, Theorem 2.2(b)]).

A mapping $f : (X, \tau) \to (Y, \sigma)$ is called ***contra-continuous*** [7] (resp. ***contra-semi-continuous*** [8], ***contra-precontinuous*** [17]) if $f^{-1}(V) \in c(X, \tau)$ (resp. $f^{-1}(V) \in SC(X, \tau)$, $f^{-1}(V) \in PC(X, \tau)$) for every $V \in \sigma$.

3. α-PRECONTINUOUS MAPPINGS

In [4] the authors gave several characterizations of α-precontinuity. In the following theorem we offer another characterizations of this type of continuity.

Theorem 1. *For a mapping $f : (X, \tau) \to (Y, \sigma)$ the following statements are eqivalent:*

(1) f is α-precontinuous.

(2) $\mathrm{pcl}_\tau(f^{-1}(B)) \subset f^{-1}(cl_{\sigma^a}(B))$ *for every subset* $B \subset Y$.

(3) $f(\mathrm{pcl}_\tau(A)) \subset cl_{\sigma^a}(f(A))$ *for every subset* $A \subset X$.

(4) $f^{-1}(\mathrm{int}_{\sigma^a}(S)) \in \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(S)))$ *for every subset* $S \subset Y$.

(5) $f^{-1}(\mathrm{int}_{\sigma^a}(S)) \subset \mathrm{pint}_\tau(f^{-1}(S))$ *for every subset* $S \subset Y$.

Proof. (1) ⇒ (2). Let f be α-precontinuous. Then $\mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(B))) \subset f^{-1}(\mathrm{cl}_{\sigma^a}(B))$ for every subset $B \subset Y$ [4, Theorem 3.1(f)]. Our inclusion follows from [2, Theorem 1.5(e)].

(2) ⇒ (1). Obvious by [2, Theorem 1.5(e)] and [4, Theorem 3.1(f)].

(1) ⇔ (3). This follows immediately from [4, Theorem 3.1(g)] and [2, Theorem 1.5(e)].

(1) ⇔ (4). Let S be any subset of Y and let $B = Y \setminus S$. Utilizing [4, Theorem 3.1(f)] we calculate as follows: $\mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(B))) \subset f^{-1}(\mathrm{cl}_{\sigma^\alpha}(B))$ iff $X \setminus f^{-1}(\mathrm{cl}_{\sigma^\alpha}(B)) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(X \setminus f^{-1}(B)))$ iff $f^{-1}(\mathrm{int}_{\sigma^\alpha}(S) \in \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(S)))$.

(4) ⇔ (5). Apply [2, Theorem 1.5(f)].

The following result has been obtained by Beceren and Noiri.

Theorem 2. [4, Theorem 3.2]. *Let* (X, τ) *and* (Y, σ) *be arbitrarily chosen spaces and let the graph mapping* $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ *for an* $f : (X, \tau) \to (Y, \sigma)$ *be given via* $g(x) = (x, f(x))$ *for each* $x \in X$. *If* g *is α-precontinuous then* f *is α-precontinuous.*

For a.c.S., a.c.H., and w.c. [35] mappings, theorems of the above type are reversible; see respectively [23, Theorems 1&2], [35, Theorem 1]. Also, in the cases of a.α.c. [41], s.w.c. [42], p.a.α.c. [11], and p.s.w.c. [11] mappings, reversibilities of this kind are possible; see [11].

We are to show that under a certain condition imposed on (X, τ) and (Y, σ), the converse of Theorem 2 holds.

We offer another proof of the following assertion.

Lemma 1. [13, p. 136]. *Let* (X, τ) *be a space,* $U \in \tau^\alpha$ *and* $V \in \mathrm{PO}(X, \tau)$. *Then* $U \cap V \in \mathrm{PO}(X, \tau)$.

Proof. We have $U \cap V \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(U))) \cap \mathrm{int}(\mathrm{cl}(V)) \subset \mathrm{int}(\mathrm{cl}(U)) \cap \mathrm{int}(\mathrm{cl}(V))$. From [39, Lemma 3.5] we infer that $U \cap V \in \mathrm{PO}(X, \tau)$.

Definition 2. *Let* (X, τ) *and* (Y, σ) *be spaces. Then*

$$(\tau \times \sigma)^\alpha = (\tau^\alpha \times \sigma^\alpha)^\alpha$$

Proof. First, we will show the inclusion

(1) $\tau^\alpha \times \sigma^\alpha \subset (\tau \times \sigma)^\alpha$.

Let $S \in \tau^\alpha \times \sigma^\alpha$. Then $S = \cup_{i \in J} S_i^1 \times S_i^2$ where $S_i^1 \in \tau^\alpha$ and $S_i^1 \in \sigma^\alpha$ for each $i \in J$. So,

$$S \subset \bigcup_{i \in J}\left(\mathrm{int}_\tau\left(\mathrm{cl}_\tau\left(\mathrm{int}_\tau\left(S_i^1\right)\right)\right) \times \mathrm{int}_\sigma\left(\mathrm{cl}_\sigma\left(\mathrm{int}_\sigma\left(S_i^2\right)\right)\right)\right)$$

$$= \bigcup_{i \in J}\left(\mathrm{int}_{\tau\times\sigma}\left(\mathrm{cl}_{\tau\times\sigma}\left(\mathrm{int}_{\tau\times\sigma}\left(S_i^1 \times S_i^2\right)\right)\right)\right) \subset \mathrm{int}_{\tau\times\sigma}\left(\mathrm{cl}_{\tau\times\sigma}\left(\mathrm{int}_{\tau\times\sigma}(S)\right)\right)$$

Therefore, $S \in (\tau \times \sigma)^\alpha$. The next inclusion to be shown is

(2) $(\tau \times \sigma)^\alpha \subset (\tau^\alpha \times \sigma^\alpha)^\alpha$.

Let $W \in (\tau \times \sigma)^\alpha$. Then we have

$$W \subset \operatorname{int}_{\tau\times\sigma}\left(cl_{\tau\times\sigma}\left(\operatorname{int}_{\tau\times\sigma}(W)\right)\right) \subset \operatorname{int}_{\tau^\alpha\times\sigma^\alpha}\left(cl_{\tau\times\sigma}\left(\operatorname{int}_{\tau\times\sigma}(W)\right)\right).$$

By [14, Lemma 1(i)] and (1) we get

$$W \subset \operatorname{int}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{cl}_{(\tau\times\sigma)^\alpha}\left(\operatorname{int}_{\tau\times\sigma}(W)\right)\right) \subset \operatorname{int}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{cl}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{int}_{\tau^\alpha\times\sigma^\alpha}(W)\right)\right).$$

Thus, $W \in \left(\tau^\alpha \times \sigma^\alpha\right)^\alpha$. We shall show now that

(3) $\left(\tau^\alpha \times \sigma^\alpha\right)^\alpha \subset (\tau \times \sigma)^\alpha$.

Let $V \in \left(\tau^\alpha \times \sigma^\alpha\right)^\alpha$. Hence $V \subset \operatorname{int}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{cl}_{\tau^\alpha\times\sigma^\alpha}\left(\operatorname{int}_{\tau^\alpha\times\sigma^\alpha}(V)\right)\right)$. By (1) and [14, Lemma 1(i)] we obtain

$$V \subset \operatorname{int}_{(\tau\times\sigma)^\alpha}\left(\operatorname{cl}_{(\tau^\alpha\times\sigma^\alpha)^\alpha}\left(\operatorname{int}_{\tau^\alpha\times\sigma^\alpha}(V)\right)\right).$$

By (2) and (1) we get $V \subset \operatorname{int}_{(\tau\times\sigma)^\alpha}\left(\operatorname{cl}_{(\tau\times\sigma)^\alpha}\left(\operatorname{int}_{(\tau\times\sigma)^\alpha}(V)\right)\right)$. This shows that

$$V \in \left((\tau\times\sigma)^\alpha\right)^\alpha = (\tau \times \sigma)^\alpha;$$

see [32]. Eventually, inclusions (2) and (3) complete the proof.

Corollary 1. *Let* (X, τ) *and* (Y, σ) *be such spaces that*

(4) $\left(\tau^\alpha \times \sigma^\alpha\right)^\alpha \subset \tau^\alpha \times \sigma^\alpha$

Then $(\tau \times \sigma)^\alpha = \tau^\alpha \times \sigma^\alpha$.

Theorem 2. *Let* (X, τ), (Y, σ) *be spaces and let mappings* f *and* g *be as in Theorem* 2. *If* (4) *holds and if* f *is* α*-precontinuous, then* g *is* α*-precontinuous.*

Proof. Suppose f is α-precontinuous. Let $x \in X$ and let $W \in (\tau \times \sigma)^\alpha$ be any set containing $g(x)$. By (4) and by Corollary 1 there exist sets $U \in \tau^\alpha$, $V \in \sigma^\alpha$, such that $g(x) = (x, f(x)) \in U \times V \subset W$. Since f is α-precontinuous, there is a set $U_1 \in \mathrm{PO}(X, \tau)$ containing x such that $f(U_1) \subset V$ [4, Theorem 3.1(c)]. Thus, $f(U \cap U_1) \subset V$ where $x \in U \cap U_1 \in PO(X, \tau)$ by Lemma 1. So we obtain $g(U \cap U_1) \subset U \times V \subset W$. This shows that g is α-precontinuous.

For every ***totally disconnected*** space (X, τ) (each open set is closed), we have $\tau = \tau^\alpha$ [19, Theorem 3.3].

Corollary 2. *Let (X, τ) and (Y, σ) be such spaces that the space $(X \times Y, \tau \times \sigma)$ is totally disconnected and let $f : (X, \tau) \to (Y, \sigma)$. Then, the graph mapping g of f is α-precontinuous if and only if f is α-precontinuous.*

Theorem 3. *Let for $i = 1, 2$, (X_i, τ_i) be arbitrary spaces and (Y_i, σ_i) be spaces fulfilling the condition $(\sigma_1^\alpha \times \sigma_2^\alpha)^\alpha \subset \sigma_1^\alpha \times \sigma_2^\alpha$. Then, mappings $f_i : (X_i, \tau_i) \to (Y_i, \sigma_i)$, $i = 1, 2$, are α-precontinuopus if and only if the product mapping $f : (X_1 \times X_2, \tau_1 \times \tau_2) \to (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$, defined via $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for each $(x_1, x_2) \in X_1 \times X_2$, is α-precontinuous.*

Proof. Sufficiency. [4, Theorem 3.4].

Necessity. Let f_i be α-precontinuous, $i = 1, 2$, and let $W \in (\sigma_1 \times \sigma_2)^\alpha$ By Corollary 1 we have $W \in \sigma_1^\alpha \times \sigma_2^\alpha$, and so $W = \cup_{j \in J} W_1^j \times W_2^j$ where $W_1^j \in \sigma_1^\alpha$, $W_2^j \in \sigma_2^\alpha$ for each $j \in J$. Using [4, Theorem 3.1(d)], for any chosen $j \in J$ we calculate as follows:

$$f^{-1}(W_1^j \times W_2^j) = f_1^{-1}(W_1^j) \times f_2^{-1}(W_2^j) \subset \mathrm{int}_{\tau_1}\left(\mathrm{cl}_{\tau_1}\left(f_1^{-1}(W_1^j)\right)\right) \times \mathrm{int}_{\tau_2}\left(\mathrm{cl}_{\tau_2}\left(f_2^{-1}(W_2^j)\right)\right)$$

$$= \mathrm{int}_{\tau_1 \times \tau_2}\left(\mathrm{cl}_{\tau_1 \times \tau_2}\left(f^{-1}(W_1^j \times W_2^j)\right)\right) \subset \mathrm{int}_{\tau_1 \times \tau_2}\left(\mathrm{cl}_{\tau_1 \times \tau_2}\left(f^{-1}(W)\right)\right).$$

Therefore, $f^{-1}(W) \subset \mathrm{int}_{\tau_1 \times \tau_2}\left(\mathrm{cl}_{\tau_1 \times \tau_2}\left(f^{-1}(W)\right)\right)$ and consequently f is α-precontinuous.

A subset S of a space (X, τ) is said to be ***simply open*** [6], if $S = O \cup N$ where $O \in \tau$ and N is nowhere dense in (X, τ). A mapping $f : (X, \tau) \to (Y, \sigma)$ is called ***simply continuous*** [6] if the preimage $f^{-1}(G)$ is simply open in (X, τ) for each $G \in \sigma$.

Each semi-open subset of a space (X, τ) is simply open in (X, τ) [21, Theorem 7]. Thus, any semi-continuous mapping is simply continuous. The converse is not the truth [6, Example 1.1.2].

Lemma 3. *If a mapping $f : (X, \tau) \to (Y, \sigma)$ is* α-precontinuous and contra-continuous, then *$f^{-1}(S) \in PC(X, \tau)$ for every simply open subset S of (Y, σ).*

Proof. Let $S = O \cup N$ be simply open in (Y, σ) ($O \in \tau$ and N is nowhere dense). Then, $f^{-1}(S) = f^{-1}(O) \cup f^{-1}(N)$ where $U_1 = f^{-1}(O) \in c(X, \tau)$ and $U_2 = f^{-1}(N) \in PC(X, \tau)$ [4, Theorem

3.7]. Applying a dual equality to that of [39, emma 3.5] we obtain $f^{-1}(S) = U_1 \cup U_2 \supset \mathrm{cl}_\tau(\mathrm{int}_\tau(U_1)) \cup \mathrm{cl}_\tau(\mathrm{int}_\tau(U_2) = \mathrm{cl}_\tau(\mathrm{int}_\tau(U_1 \cup U_2)) = \mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(S)))$. Therefore $f^{-1}(S) \in \mathrm{PC}(X, \tau)$.

Theorem 4. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \nu)$ be given mappings. If g is simply continuous, f is α-precontinuous and contra-continuous, then $g \circ f : (X, \tau) \to (Z, \nu)$ is contra-precontinuous.*

Proof. It follows from Lemma 3.

Theorem 5. *Let a mapping $f : (X, \tau) \to (Y, \sigma)$ be α-precontinuous, continuous, contra-semicontinuous, and let (Y, σ) be a T_1-space. Then, for each $y \in Y$ and for each $V \in \sigma$ such that $y \in \mathrm{int}_\sigma(\mathrm{cl}_\sigma(V))$ we have $f^{-1}(V \cup \{y\}) \in \mathrm{RO}(X, \tau)$.*

Proof. Let $V \in \sigma$ be arbitrarily chosen. Remark that if $y \in V$, then up to continuity and contra-semicontinuity of f we get $f^{-1}(V) \in \mathrm{RO}(X, \tau)$. By α-precontinuity of f [4, Theorem 3.8] for every $y \in \mathrm{int}_\sigma(\mathrm{cl}_\sigma(V))$ we have $f^{-1}(V \cup \{y\}) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(V \cup \{y\}))) = \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(V)) \cup \mathrm{cl}_\tau(f^{-1}(\{y\}))) \subset \mathrm{cl}_\tau(f^{-1}(\{y\})) \cup \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(V)))$ [2, Lemma 1.1.(b)]. Since f is continuous and (Y, σ) is a T_1-space, $\mathrm{cl}_\tau(f^{-1}(\{y\})) = f^{-1}(\{y\})$. From contra-semicontinuity of f we infer that $f^{-1}(\{y\}) \cup \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(V))) \subset f^{-1}(V \cup \{y\})$. Hence clearly, $f^{-1}(V \cup \{y\}) \in \mathrm{RO}(X, \tau)$.

Remark 1. Continuity and α-precontinuiuty are independent of each other [4, Examples 2.1 & 2.2]

Theorem 6. *Assume an $f : (X, \tau) \to (Y, \sigma)$ has the following : for each $y \in Y$ and each $V \in \sigma$ with $y \in \mathrm{int}_\sigma(\mathrm{cl}_\sigma(V))$, the preimage $f^{-1}(V \cup \{y\})$ is in RO (X, τ). Then, f is α-precontinuous, continuous, and contra-semicontinuous.*

Proof. The α-precontinuity of f is clear by [4, Theorem 3.8]. Let $V \in \sigma$ be arbitrary and let $y \in V$. By $f^{-1}(V) \in \mathrm{RO}(X, \tau)$ it follows evidently that f is continuous and contra-semicontinuous.

4. PRECONTINUOUS MAPPINGS

It is known that each α-precontinuous mapping is precontinuous and that the converse is not true, in general [4, Remark]. Thus, results concerning precontinuous mappings hold also in the α-precontinuity case. Some of them are not mentioned in [4] the reader is advised to see for instance [29, Theorems 2.3 & 2.4 & 2.5].

We recall that for every two a.c.S. mappings f_1, f_2 from a space (X, τ) into a Hausdorff space (Y, σ), the set $\{x \in X : f_1(x) = f_2(x)\}$ is closed in (X, τ) [24, Theorem 4]. This result is obviously true for any two mappings f_1, f_2 with a stronger type of continuity than a.c.S. (see for instance [40, Diagram p. 249]), in particular for ***completely continuous*** mappings [3]. On the other hand, the following is evident.

Remark 2. A mapping $f : (X, \tau) \to (Y, \sigma)$ is precontinuous and contra-semicontinuous if and only if it is completely continuous.

Theorem 7. *Let (X, τ) be arbitrary, (Y, σ) be Hausdorff, and let mappings $f_1, f_2 : (X, \tau) \to (Y, \sigma)$ be given. If f_1 is α-continuous and f_2 is precontinuous, then $A = \{x \in X : f_1(x) = f_2(x)\} \in \mathrm{PC}(X, \tau)$.*

Proof. Let x be any point of $X \setminus A$. Hence $f_1(x) \neq f_2(x)$ and since (Y, σ) is Hausdorff, there exist sets $V_1, V_2 \in \sigma$ such that $f_1(x) \in V_1$, $f_2(x) \in V_2$, and $V_1 \cap V_2 = \emptyset$.

By Lemma 1 we have $x \in U_x = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \in \mathrm{PO}(X, \tau)$. We will show that $U_x \subset X \setminus A$. Suppose not. There exists a point $x' \in U_x$ such that $x' \in A$. Hence $f_1(x') = f_2(x') \in V_2$. This implies that $x' \in f_1^{-1}(V_1) \cap f_1^{-1}(V_2) = \emptyset$, a contradiction. Thus, the set $X \setminus A$ is preopen and consequently $A \in \mathrm{PC}(X, \tau)$.

Remark 3. (a) [25, Example 1] and [48, Example 2.1] show that precontinuity and a.c.S. are independent notions [23, p. 413].

(b) [39, Examples 3.9 & 3.10] show that α-continuity and a.c.S. are independent of each other.

An analysis of the proof of Theorem 7 leads to the following slight improvement of [15, Theorem 2.6(1)]. We use the fact that in any (X, τ), if $U \in \tau^\alpha$ and $V \in \mathrm{SO}(X, \tau)$, then $U \cap V \in \mathrm{SO}(X, \tau)$ [32].

Theorem 8. *Let (X, τ) be arbitary, (Y, σ) be Hausdorff, and $f_1, f_2 : (X, \tau) \to (Y, \sigma)$. If f_1 is α-continuous and f_2 is semi-continuous, then the set $A = \{x \in X : f_1(x) = f_2(x)\} \in \mathrm{SC}(X, \tau)$.*

Remark 4. (a) [21, Example 8] and [38, Example 4.1] show that semi-continuity and a.c.S. are independent of each other.

(b) [37, Examples 2.3] shows that there exists an α-continuous mapping which is not continuous. Obviously, each continuous map is α-continuous.

(c) [31, Examples 3.1 & 3.2] show that semi-continuity and precontinuity are independent of each other.

If mappings $f_1, f_2 : (X, \tau) \to (Y, \sigma)$ are both α-continuous (hence precontinuous), but the space (Y, σ) is not Hausdorff, then the set A from Theorems 7 and 8 mut not be even semi-preclosed in (X, τ). It is worth to see also [39, Theorem 4.9].

Example 1. *Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}\}$, and $\sigma = \{\emptyset, Y, \{a\}\}$. Define*

$f_1, f_2 : (X, \tau) \to (Y, \sigma)$ *as follows:* $f_1(a) = b$, $f_2(a) = c$, $f_1(b) = f_2(b) = f_1(c) = f_2(c) = a$. *Then* f_1 *and* f_2 *are* α-*continuous and the set* $\{x \in X : f_1(x) = f_2(x)\} = \{b, c\} \notin \mathrm{SPC}(X, \tau)$.

Using Theorem 8 and [15, Theorem 2.4] we slightly improve [15, Theorem 2.6(2)].

Corollary 3. *Let* (X, τ) *be arbitrary,* (Y, σ) *be Hausdorff, and* $f_1, f_2 : (X, \tau) \to (Y, \sigma)$. *If* f_1 *is* α-*continuous,* f_2 *is semi-continuous, and* $f_1 = f_2$ *on a dense subset of* (X, τ), *then* $f_1 = f_2$ *on* X.

Definition 1. *A subset* S *of a space* (X, τ) *is said to be* **p-dense** *in* (X, τ) *if* $\mathrm{pcl}_\tau(S) = X$.

It is obvious that every subset p-dense in (X, τ) is dense in (X, τ), but the converse is not always true.

Example 2. *Consider the space* **R** *of all reals with Euclidean topology* τ_e, $S = \mathbf{Q}$ (**Q** *the set of all rationals*). *By* [2, Theorem 1.5(e)] *we have* $\mathrm{pcl}_{\tau_e}(S) = S$.

The next result follows immediately from Theorem 7.

Corollary 4. *Let* (X, τ) *be arbitrary,* (Y, σ) *be Hausdorff, and* $f_1, f_2 : (X, \tau) \to (Y, \sigma)$. *If* f_1 *is* α-*continuous,* f_2 *is precontinuous, and* $f_1 = f_2$ *on a* **p**-*dense subset of* (X, τ), *then* $f_1 = f_2$ *on* X.

Theorem 9. *Let* (X, τ) *be arbitrary and* $S \in \mathrm{SO}(X, \tau)$. *Then* S *is* p-*dense in* (X, τ) *if and only if* S *is dense in* (X, τ).

Proof. Sufficiency. Apply [2, Theorem 1.5(e)] and [34, Lemma 2].

Theorem 10. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a precontinuous surjection. If a set* $S \in \mathrm{SO}(X, \tau)$ *is dense in* (X, τ), *then* $f(S)$ *is dense in* (Y, σ).

Proof. [20, Proposition 3.1]. The reader is advised to compare the characterization (3) of precontinuous mapings given in [47, Theorem 6].

Theorem 11. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an* α-*precontinuous surjection. If a set* $S \subset X$ *is* p-*dense in* (X, τ), *then* $f(S)$ *is dense in* (Y, σ).

Proof. This follows from Theorem 1(3).

A pace (X, τ) is called a ***D-space*** [22] if each nonempty set $V \in \tau$ is dense in (X, τ). By [22, Theorem 1] and [49, Theorem 17], being a $\mathcal{D}$-space, ***S-connectedness*** (X connot be expressed as the union of two nonempty disjoint semi-open subsets), and ***irreducibility*** are equivalent.

A space (X, τ) is said to be ***submaximal*** if for every dense subset S of (X, τ) we have $S \in \tau$.

Theorem 12. *Let* (X, τ) *be a* $\mathcal{D}$*-space and* (Y, σ) *be submaximal. If a surjection* $f : (X, \tau) \to (Y, \sigma)$ *is precontinuous then* f *is open.*

Proof. [20, Proposition 3.1] or [47, Theorem 6].

Mashour et al. [28, theorem 1] established the following characterization of precontinuous mappings: $f : (X, \tau) \to (Y, \sigma)$ is precontinuous if and only if $f(\text{cl}_\tau(\text{int}_\tau(U))) \subset \text{cl}_\sigma(f(U))$ for every $U \subset X$. By this and by [2, Theorem 1.5(e)] one easily obtains the following.

Lemma 4. *Let* (X, τ) *and* (Y, σ) *be any spaces. A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is precontinuous if and only if* $f(\text{pcl}_\tau(U) \subset \text{cl}_\sigma(f(U))$ *for every* $U \subset X$.

Theorem 13. *Let* $f : (X_i, \tau_i) \to (Y_i, \sigma_i)$, $i = 1, 2$, *be* (X, τ) *be precontinuous surjections. If a set* $S_1 \times S_2 \subset X_1 \times X_2$ *is* p*-dense in* $(X_1 \times X_2, \tau_1 \times \tau_2)$ *then the product's image* $f(S_1 \times S_2)$ *is dense in* $(Y_1 \times Y_2, \sigma_1 \times \sigma_2)$.

Proof. Let a set $S_1 \times S_2 \subset X_1 \times X_2$ be p-dense in $X_1 \times X_2$. Utilizing [12, Lemma 5.2] and Lemma 4 we get $Y_1 \times Y_2 = f(\text{pcl}(S_1 \times S_2)) \subset f\left(\text{pcl}_{\tau_1}(S_1) \times \text{pcl}_{\tau_2}(S_2)\right) = f_1\left(\text{pcl}_{\tau_1}(S_1)\right)$

$\times f_2\left(\text{pcl}_{\tau_2}(S_2)\right) \subset \text{cl}_{\sigma_1}\left(f_1(S_1)\right) \times \text{cl}_{\sigma_2}\left(f_2(S_2)\right) = \text{cl}_{\sigma_1 \times \sigma_2}\left(f(S_1 \times S_2)\right).$

Theorem 14. *Let* (X_i, τ_i) *be* $\mathcal{D}$*-spaces and* (Y_i, σ_i) *be submaximal,* $i = 1, 2$. *If mappings* $f : (X_i, \tau_i) \to (Y_i, \sigma_i)$ *are precontinuous surjections then the product mapping* f *is open.*

Proof. Without difficulties we infer from [28, Theorem 1] that $f(U_1 \times U_2) \in \sigma_1 \times \sigma_2$ for any $U_1 \in \tau$, $U_2 \in \tau$; a calculation is similar to that in the proof of Theorem 13.

A space (X, τ) is said to be β-***connected*** [44] if X cannot be expressed as the union of two nonempty disjoint semi-preopen subsets of (X, τ). A space (X, τ) is β-disconnected if it is not β-connected.

Definiton 2. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* $\mathcal{P}$***-open*** *if* $f(V) \in \sigma$ *for each* $V \in$ PO(X, τ).

Recall that Jankovic [20] calls a mapping p-***open*** if it preserves preopen sets. Obviously each $\mathcal{P}$-open mapping is open, but these concepts are strictly distinct.

Example 3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, \{a, b, c\}, \{a\}, \{b, c\}\}$. *The identity mapping* id : $(X, \tau) \to (X, \tau)$ *is open but not* $\mathcal{P}$*-open, because* id $(\{b\}) = \{b\} \notin \tau$.

Theorem 15. *Let* (X, τ) *be* β*-connected and* (Y, σ) *be submaximal. If* $f : (X, \tau) \to (Y, \sigma)$ *is a precontinuous surjection then it is* $\mathcal{P}$*-open.*

Proof. We use [18, Theorem 3.1(2)] and Lemma 4.

We complete characterizations of β-connected spaces given in [18, Theorem 3.1] (see also [43]) with the following.

Theorem 16. *For any space* (X, τ) *the following are equivalent:*

(1) (X, τ) *is* β*-connected;*

(2) $\operatorname{pint}(\operatorname{pcl}(V)) = X$ *for each nonempty* $V \in PO(X, \tau)$*;*

(3) $\operatorname{spint}(\operatorname{pcl}(V)) = X$ *for each nonempty* $V \in SPO(X, \tau)$*;*

(4) $\operatorname{pint}(\operatorname{spcl}(V)) = X$ *for each nonempty* $V \in PO(X, \tau)$*;*

(5) $\operatorname{spint}(\operatorname{spcl}(V)) = X$ *for each nonempty* $V \in SPO(X, \tau)$.

Proof. Use respective parts of [18, Theorem 3.1]

Lemma 5. *In every topological space* (X, τ) *and for any* $W \subset X$ *we have*

(a) $\operatorname{pcl}_\tau(\operatorname{pint}_\tau(W)) \in SPO(X, \tau)$,

(b) $\operatorname{pcl}_\tau(\operatorname{spint}_\tau(W)) \in SPO(X, \tau)$,

(c) $\operatorname{spcl}_\tau(\operatorname{pint}_\tau(W)) \in SPO(X, \tau)$.

Proof. (a) By [2, Theorem 1.5(e)] we have what follows.

$$\begin{aligned} \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pcl}(\operatorname{pint}(W))))) &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pint}(W) \cup \operatorname{cl}(\operatorname{int}(\operatorname{pint}(W)))))) \\ &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pint}(W)))) \cup \operatorname{cl}(\operatorname{int}(\operatorname{pint}(W))) \\ &\supset \operatorname{pint}(W) \cup \operatorname{cl}(\operatorname{int}(\operatorname{pint}(W))) = \operatorname{pcl}(\operatorname{pint}(W)), \end{aligned}$$

because $\operatorname{pint}(W) \in PO(X, \tau)$.

(b) By [2, Theorem 1.5(e)] we get

$$\begin{aligned} \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pcl}(\operatorname{spint}(W))))) &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{spint}(W) \cup \operatorname{cl}(\operatorname{int}(\operatorname{spint}(W)))))) \\ &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{spint}(W)))) \cup \operatorname{cl}(\operatorname{int}(\operatorname{spint}(W))) \\ &\supset \operatorname{spint}(W) \cup \operatorname{cl}(\operatorname{int}(\operatorname{spint}(W))) = \operatorname{pcl}(\operatorname{spint}(W)). \end{aligned}$$

(c) Applying [2, Theorem 2.15] we obtain

$$\begin{aligned} \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{spcl}(\operatorname{pint}(W))))) &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pint}(W) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{pint}(W))))))) \\ &= \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\operatorname{pint}(W)))) \cup \operatorname{cl}(\operatorname{int}(\operatorname{pint}(W)))) \\ &\supset \operatorname{pint}(W) \cup \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{pint}(W)))) = \operatorname{spcl}(\operatorname{pint}(W)). \end{aligned}$$

Theorem 17. *Let* (X, τ) *be* β*-disconnected. Then, the space* X *allows the following partitions* [9, p.13]:

(a) $\{\text{pint}(\text{pcl}(S)), \text{pcl}(\text{pint}(X \setminus S))\} \subset \text{SPO}(X, \tau)$ *for a certain nonempty* $S \in \text{PO}(X, \tau)$,

(b) $\{\text{pint}(\text{spcl}(S)), \text{pcl}(\text{spint}(X \setminus S))\} \subset \text{SPO}(X, \tau)$ *for a certain nonempty* $S \in \text{PO}(X, \tau)$,

(c) $\{\text{spint}(\text{pcl}(S)), \text{spcl}(\text{pint}(X \setminus S))\} \subset \text{SPO}(X, \tau)$ *for a certain nonempty* $S \in \text{SPO}(X, \tau)$,

(d) $\{\text{spint}(\text{spcl}(S)), \text{spcl}(\text{spint}(X \setminus S))\} \subset \text{SPO}(X, \tau)$ *for a certain nonempty* $S \in \text{SPO}(X, \tau)$.

Proof. (a) From β-disconnectedness of (X, τ) and from Theorem 16(2') we infer that there exists a nonempty $S \in \text{PO}(X, \tau)$ such that $U_1 = \text{pint}(\text{pcl}(S)) \neq X$. Obviously $U_1 \in \text{SPO}(X, \tau)$. We shall show that $U_1 \neq \emptyset$. Suppose not. By [2, Theorem 1.5(f)] We get $\emptyset = \text{pcl}(S) \cap \text{int}(\text{cl}(\text{pcl}(S)))$. Applying [39, Lemma 3.5] Lemma 3.5] we have $\emptyset = \text{int}(\text{cl}(\text{pcl}(S))) \supset S$ and so $S = \emptyset$. A contradiction. Put now $U_2 = X \setminus \text{pint}(\text{pcl}(S)) = \text{pcl}(\text{pint}(X \setminus S))$. Clearly, $X \neq U_2 \neq \emptyset$ and by Lemma 5(a), $U_2 \in \text{SPO}(X, \tau)$.

Proofs for (b) – (d) are similar to the above. We use respective parts of Theorem 16, Lemma 5, and [18, Lemma 3.1].

(b) We shall show only that $U_1 = \text{pint}(\text{spcl}(S)) \neq \emptyset$, where a nonempty $S \in \text{PO}(X, \tau)$ is such that $U_1 \neq X$. Suppose not. By [2, Theorem 1.5(f)] and [39, Lemma 3.5] we obtain $\emptyset = \text{spcl}\ (S) \cap \text{int}(\text{cl}(\text{spcl}(S))) = \text{int}(\text{cl}(\text{spcl}(S)))$. So, $\emptyset = \text{cl}(\text{int}(\text{cl}(\text{spcl}(S)))) \supset \text{spcl}\ (S)$ [18, Lemma 3.1(2)], a contradiction.

(c) We shall show only that $U_1 = \text{spint}\ (\text{pcl}(S)) \neq \emptyset$, where a nonempty $S \in \text{SPO}(X, \tau)$ is such that $U_1 \neq X$. Suppose not. By [2, Theorem 3.21(a)] we have $\emptyset = \text{spint}\ (\text{pcl}(S)) \supset \text{spint}\ (\text{spcl}(S)) \supset S$, a contradiction.

(d) Let $U_1 = \text{spint}\ (\text{spcl}(S))$, where a nonempty $S \in \text{SPO}(X, \tau)$ is such that $U_1 \neq X$. Suppose $U_1 = \emptyset$. Then $\emptyset = \text{spint}(\text{spcl}(S)) \supset S$ [2, Theorem 3.21(a)], a contradiction.

Corollary 5. *If an* $f : (X, \tau) \to (Y, \sigma)$ *is a not* $\mathcal{P}$*-open precontinuous surjection and* (Y, σ) *is submaximal, then* X *allows the partitions* (a) – (d) *from Theorem* 17.

Proof. By Theorem 15.

Definition 3. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is said to be* ***pre-α-open*** *if* $f(S) \in \sigma^\alpha$ *for each* $S \in \tau^\alpha$.

Theorem 18. *Let a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be open and precontinuous. Then it is pre-α-open.*

Proof. Let a set $A \in \tau^\alpha$ be arbitrarily chosen. By [39, Lemma 4.12(1)] there exists a $U \in \tau$ such that $U \subset A \subset \text{int}_\tau(\text{cl}_\tau(U))$. Since f is open, it follows from [36, Lemma 1.4] that

$f(\mathrm{int}_\tau(\mathrm{cl}_\tau(U))) \subset \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(U))))) = \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U)))$. Since f is precontinuous, $\mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U))) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(U)))$ [47, Theorem 6.(3)]. Finally, we obtain $f(U) \subset f(A) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(U)))$ and hence, by [39, Lemma 4.12(1)], $f(A) \in \sigma^\alpha$.

The notions of openness and pre-α-openness are independent of each other.

Example 4. (a) *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, *and* $\sigma = \{\emptyset, X, \{a\}, \{b, c\}\}$. *Then, the identity mapping* $\mathrm{id} : (X, \tau) \to (X, \sigma)$ *is open, but it is not pre-α-open since* $\mathrm{id}(\{a, b\}) \notin \sigma^\alpha$.

(b) *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, *and* $\sigma = \{\emptyset, X, \{a\}\}$. *Then, the identity mapping* $\mathrm{id} : (X, \tau) \to (X, \sigma)$ *is pre-α-open and not open since* $\mathrm{id}(\{a, b\}) \notin \sigma$.

A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be ***weakly open*** [46] (resp. ***preopen*** [28]) if $f(U) \subset \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U)))$ (resp. $f(U) \in \mathrm{PO}(Y, \sigma)$) for every set $U \in \tau$. Each pre-α-open mapping is preopen, but the converse doesn't hold (Example 4(a)).

Theorem 19. *If* $f : (X, \tau) \to (Y, \sigma)$ *is weakly open and precontinuous, then it is preopen.*

Proof. Let $U \in \tau$ be arbitrary. By [47, Theorem 6(3)] we have $f(U) \subset \mathrm{int}_\sigma(f(\mathrm{cl}_\tau(U))) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(f(U)))$.

In [47, Theorem 11], it is shown that preopenness and the so-called ***a.o.W.*** property [50] are equivalent notions. Recall that weak openness and a.o.W. are independent of each other [36, p. 315].

5. α-IRRESOLUTNESS OF MAPPINGS

A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be ***semi-open*** [5] (resp. ***almost open in the sense of Singal and Singal*** [48] or briefly ***a.o.S.***) if $f(U) \in \mathrm{SO}(Y, \sigma)$ (resp. $f(U) \in \sigma$) for every set $U \in \tau$ (resp. $U \in \mathrm{RO}(X, \tau)$). The concepts of preopenness, semi-openness, and a.o.S. are pairwise independent [36].

Lemma 6. *If an* $f : (X, \tau) \to (Y, \sigma)$ *is preopen and precontinuous, then for each set* $S \in \sigma^\alpha$ *there exists a* $U_s \in \sigma$ *with*

$$f^{-1}(U_s) \subset f^{-1}(S) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))). \tag{5}$$

Proof. Let an $S \in \sigma^\alpha$ be arbitrarily chosen. By [39, Lemma 4.12(1)] there exists a set $U_s \in \sigma$ such that $U_s \subset S \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))$. Since f is precontinuous, using [47, Theorem 11] we obtain

$$f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{cl}_\sigma(U_s)))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))).$$

This completes the proof.

Lemma 7. *If an* $f : (X, \tau) \to (Y, \sigma)$ *is semi-open and precontinuous, then for each set* $S \in \sigma^\alpha$ *thee exists a* $U_s \in \sigma$ *that satisfies* (5).

Proof. Let an $S \in \sigma^\alpha$. By [39, Lemma 4.12(1)] there exists a set $U_s \in \sigma$ such that $U_s \subset S \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))$. Since f is precontinuous, we have

$$f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))))).$$

It follows by [39, Lemma 4.14] (or by [2, Theorem 1.5(a)]) that $f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{scl}_\sigma(U_s))))$. From [33, Theorem 2] we infer that $f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))$. Inclusions $f^{-1}(U_s) \subset f^{-1}(S) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))$ complete the proof.

Lemma 7 slightly improves a respective part of the proof of [39, Theorem 4.16].

Lemma 8. *If a bijection* $f : (X, \tau) \to (Y, \sigma)$ *is a.o.S. and precontinuous, then for each set* $S \in \sigma^\alpha$ *there exists a* $U_s \in \sigma$ *that satisfies* (5).

Proof. Let an $S \in \sigma^\alpha$. There exists a set $U_s \in \sigma$ such that $U_s \subset S \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))$. We have $f^{-1}(U_s) \subset \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))))$ since by hypothesis $f^{-1}(U_s) \in \mathrm{PO}(X, \tau) \subset \mathrm{SPO}(X, \tau)$. Put $F = Y \setminus f(X \setminus \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))))$. Hence $f(X \setminus f^{-1}(U_s)) \supset f(X \setminus \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))))) = Y \setminus F$. It implies that $F \supset Y \setminus (f(X)\setminus U_s) = U_s$. The set F is closed in (Y, σ), because $\mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))) \in \mathrm{RC}(X, \tau)$ and f is a.o.S. Thus, $\mathrm{cl}_\sigma(U_s) \subset F$. Furthermore, we have $f^{-1}(F) = \mathrm{cl}_\tau(\mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))))$. Finally, we obtain $f^{-1}(U_s) \subset f^{-1}(S) \subset f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma((U_s))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{int}_\sigma(\mathrm{cl}_\sigma(U_s))))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(\mathrm{cl}_\sigma(U_s)))) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(F))) = \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s)))$. This completes the proof.

Every α-irresolute mapping is semi-continuous but even a continuous mapping must not be α-irresolute [27, Example 1]. Every α-irresolute mapping is precontinuous and the converse is not necessarily true [4, Remark].

Theorem 20. *Let a mapping* $f : (X, \tau) \to (Y, \sigma)$ *be precontinuous, semi-continuous (equiv. α-continuous* [39, Theorem 3.2]), *and let f be either*

(a) *preopen* or

(b) *semi-open* [39, Theroem 4.16].

Then f is α-irresolute.

Proof. Consider by turns the inclusions (5) from Lemmas 6, 7:

$$f^{-1}(U_s) \subset f^{-1}(S) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(f^{-1}(U_s))),$$

where $S \in \sigma^\alpha$ and $U_s \in \sigma$. Since f is semi-continuous, $\mathrm{cl}_\tau(f^{-1}(U_s)) = \mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(U_s)))$ [34, Lemma 2]. Thus

$$\mathrm{int}_\tau(f^{-1}(U_s)) \subset f^{-1}(S) \subset \mathrm{int}_\tau(\mathrm{cl}_\tau(\mathrm{int}_\tau(f^{-1}(U_s)))).$$

By [39, Lemma 4.12(1)], f is α-irresolute.

Remark 5. Obviously, Lemmas 6, 7 and Theorem 20 hold if we replace 'precontinuous' by 'α-precontinuous'. In [4, Examples 2.1 & 2.2] it has been shown that α-precontinuity and semi-continuity are independent notions.

Noiri has established that each a.o.S. and α-continuous mapping is α-irresolute [39, Theorem 4.13]. Since the proof of this result is not clear (on a certain stage), we shall prove it in a different way.

Lemma 9. *For a mapping $f : (X, \tau) \to (Y, \sigma)$ the following statements are equivalent:*

(a) *f is α-irresolute.*

(b) $f(\mathrm{cl}_{\tau^\alpha}(A)) \subset \mathrm{Cl}_{\sigma^\alpha}(f(A))$ *for each $A \subset X$.*

(c) $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}_{\sigma^\alpha}(f(A))$ *for each $A \subset X$.*

Proof. (a) $\Leftrightarrow$ (b). Obvious.

(b) $\Leftrightarrow$ (c). Apply [2, Theorem 1.5(c)].

Proof of [39, Theorem 4.13]. We have $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}(f(A))$ for each subset $A \subset X$, because f is α-continuous [30, Theorem 1.1(iv)]. Since f is a.o.S., we get

$$\mathrm{cl}(f(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}(\mathrm{int}(f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))))) \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f(A)))).$$

Utilizing [30, Theorem 1.1(iv)] once more, one easily obtains that $f(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))) \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(f(A)))) \cup f(A)$ for each $A \subset X$. So, by [2, Theorem 1.5(c)] and Lemma 9(c), f is α-irresolute.

6. HAUSDORFFNESS AND NORMALITY OF SPACES

Definition 4. *A topological space (X, τ) is said to be* **p-Hausdorff** *if for each pair of distinct points $x, y \in X$ there exist disjoint sets $U_x, U_y \in \mathrm{PO}(X, \tau)$ with $x \in U_x$ and $y \in U_y$.*

Each Hausdorff space is p-Hausdorff. The converse is false in general, as the following example shows.

Example 5. *Consider a space (X, τ) where $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$.*

The concepts of p-hausdorffiness and the so-called ***semi-Hausdorffness*** [26], are independent of each other. It is evident by Example 5 and [26, Example 4.1].

Theorem 21. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a precontinuous injection. If* (Y, σ) *is Hausdorff then* (X, τ) is p-*Hausdorff.*

Proof. Omitted.

The results we complete this section with, hold also in some other cases; in particular, for completely continuous mappings (see Remark 2 and [40, Diagram p.249]).

Theorem 22. *If* $f : (X, \tau) \to (Y, \sigma)$ *is an a.c.S. injection and* (Y, σ) *is Hausdorff, then* (X, τ) *is Hausdorff.*

Proof. Use [48, Theorem 2.2(b)] and [10, Lemma 4].

Lemma 10. *A Hausdorff space* (X, τ) *is normal if and only if for each pair of disjoint sets,* $F_1, F_2 \in c(X, \tau)$ *there exist disjoint* $U_1, U_2 \in RO(X, \tau)$ *with* $F_1 \subset U_1$ *and* $F_2 \subset U_2$.

Proof. Similar to that of [10, Lemma 4].

Theorem 23. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an a.c.S. closed injection. If* (Y, σ) *is normal then* (X, τ) *is normal.*

Proof. Use [48, Theorem 2.2(b)] and Lemma 10.

Recently, the author has introduced the concept of closed-open mappings [10, Definition 1].

Definition 5. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *will be called* **closed-α-open** *if the image* $f(F) \in \sigma^\alpha$ *for each* $F \in c(X, \tau)$.

Each closed-open mapping is closed-α-open. The converse doesn't hold.

Example 6. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$. *Define the* $f : (X, \tau) \to (X, \tau)$ *via* $f(a) = c$, $f(b) = a$, $f(c) = b$. *Then,* f *is closed-α-open but not closed-open, because* $f(\{b, c\}) \in \tau^\alpha \backslash \tau$.

Theorem 24. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be an a.c.S. closed-α-open injection. If* (Y, σ) *is Hausdorff, then* (X, τ) *is normal.*

Proof. Hausdorffness of (X, τ) is clear by Theorem 22. Let $F_1, F_2 \in c(X, \tau)$ be disjoint. We have $f(F_1) \cap f(F_2) = \emptyset$ and there exist sets $G_i \in \sigma$ such that $G_i \subset f(F_i) \subset \mathrm{int}_\sigma(\mathrm{cl}_\sigma(G_i))$, $i = 1, 2$ [39, Lemma 4.12(1)]. Since $G_1 \cap G_2 = \emptyset$, it follows that

$$\mathrm{int}_\sigma(\mathrm{cl}_\sigma(G_1)) \cap \mathrm{int}_\sigma(\mathrm{cl}_\sigma(G_2)) = \emptyset.$$

The sets $U_i = \mathrm{int}_\sigma(\mathrm{cl}_\sigma(G_i)) \in RO(Y, \sigma)$, $i = 1, 2$. Therefore, we obtain for both i's that $F_i \subset f^{-1}(U_i) \in \tau$ [48, Theorem 2.2(b)] and $f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$. This shows that (X, τ) is normal.

REFERENCES

1. M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ., **12**(1) (1983), 77-90.

2. D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38** (1986), 24-32.

3. S. P. Arya, R. Gupta, *On strongly continuous mappings*, Kyungpook Math. J., **14** (1974), 131-143.

4. Y. Beceren, T. Noiri, *On α-precontinuous functions*, Far East J. Math. Sci., Special Volume III (2000), 295-303.

5. N. Biswas, *On some mappings in topological spaces*, Bull. Calcutta Math. Soc., **61** (1969), 127-135.

6. N. Biswas, *On some mappings in topological spaces*, Thesis for D. Ph. Arts Degree, Univ. of Calcutta 1970.

7. J. Dontchev, *Contra-continuous functions and strongly S-closed spaces*, Internat. J. Math. Mth. Sci., **19**(2) (1996), 303-310.

8. J. Dontchev, T. Noiri, *contra-semicontinuous functions*, Math. Pannonica, **10**(2) (1999), 159-168.

9. J. Dugundji, *Topology*, Allyn & Bacon, Inc., Boston 1966.

10. Z. Duszyński[1], *Some remarks on almost α-continuous functions*, Kyungpook Math. J., **44**(2) (2004); 249-260.

11. Z. Duszyński, *Properties of prealmost α-continuous and presemi-weakly continuous functions*, Acta Mathematica Hungarica, **105**(3) (2004), 231-239.

12. S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour, T. Noiri, *On* p-*regular spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, **27(75)**(4) (1983), 311-315.

13. M. Ganster, *Preopen sets and resolvable spaces*, Kyungpook Math. J., **27**(2) (1987), 135-143.

14. G. L. Garg. D. Sivaraj, *Semitopological properties*, Mat. Vesnik, **36** (1984), 137-142.

15. T. R. Hamlett, *Semi continuous functions*, Math. Chronicle, **4** (1976), 101-107.

16. T. Husain, *Almost continuous mappings*, Prace Mat., **10** (1966), 1-7.

17. S. Jafari, T. Noiri, *On contra-precontinuous functions*, Bul. Malaysian Math. Sci. Soc., **25** (2002), 115-128.

18. S. Jafari, T. Noiri, *Properties of β-connected spaces*, Acta Math. Hungar., **101**(3) (2003), 227-236.

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1. The reader is requested to use the correct official spelling of author's name, i.e., Duszyński.

19. D. S. Janković, *On locally irreducible spaces,* Ann. Soc. Sci. Bruxelles, **97**(2) (1983), 59-72.

20. D. S. Janković, *A note on mappings of extremally disconnected spaces,* Acta Mathematica Hungarica, **46**(1-2) (1985), 83-92.

21. N. Levine, *Semi-open sets and semi-continuity in topological spaces,* Amer. Math. Monthly, **70** (1963), 36-41.

22. N. Levine, *Dense topologies,* Amer. Math. Monthly, **75** (1968), 847-852.

23. P. E. Long, D. A. Carnahan, *Comparing almost continuous functions,* Proc. Amer. Math. Soc., **38**(2) (1973), 413-418.

24. P. E. Long, L. L. Herrington, *Properties of almost-continuous functions,* Bolletino U.M.I., **10**(4) (1974). 336-342.

25. P. E. Long. E. E. McGehee Jr. *Properties of almost continuous funcitons,* Proc. Amer. Math. Soc., **24** (1970), 175-180.

26. S. N. Maheshwari, R. Prasad, *Some new separation axioms*, Ann. Soc. Sci. Bruxelles, **89**(3) (1975), 395-402.

27. S. N. Maheshwari, S. S. Thakur, *On α-irresolute mappings,* Tamkang J. Math., **11** (1980), 209-214.

28. A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, *On precontinuous and weak precontinuous mappings,* Proc. Math. and Phys. Soc. Egypt, **53** (1982), 47-53.

29. A. S. Mashhour, I. A. Hasanein, S. N. El-Deeb, *A note on semi-continuity and precontinuity,* Indian J. Pure Appl. Math., **13**(10) (1982), 1119-1123.

30. A. S. Mashhour, I. A. Hasanein, S. N. El-Deeb, *α-continuous and α-open mappings,* Acta Mathematica Hungarica, **41** (1983), 213-218.

31. A. Neubrunnová, *On certain generalizations of the notion of continuity,* Mat Časopis, **23**(4) (1973), 374-380.

32. O. Njåstad, *On some classes of nearly open sets,* Pacific J. Math., **15** (1965), 961-970.

33. T. Noiri, *Remarks on semi-open mappings,* Bull. Cal. Math. Soc., **65** (1973), 197-201.

34. T. Noiri, *On semi-continuous mappings,* Lincei-Rend. Sc. fis. mat. e nat., **54** (1973), 210-214.

35. T. Noiri, *On weakly continuous mappings,* Proc. Amer. Math. Soc., **46**(1) (1974), 120-124.

36. T. Noiri, *Semi-continuity and weak-continuity,* Czechoslovak Mathematical Journal, **31** (106) (1981), 314-321.

37. T. Noiri, *A function which preserves connected spaces,* Čas. pěst. mat., **107** (1982), 393-396.

38. T. Noiri, *Almost-open functions,* Indian J. Math., **25**(1) (1983), 73-79.

39. T. Noiri, *On α-continuous functions,* Čas. pěst. mat., **109** (1984), 118-126.

40. T. Noiri, *Super-continuity and some strong forms of continuity,* Indian J. Pure Appl. Math., **15**(3), (1984), 241-250.

41. T. Noiri, *Almost α-continuous functions,* Kyungpook Math. J, **28**(1) (1988), 71-77.

42. T. Noiri, B. Ahmad, *On semi-weakly continuous mappings,* Kyungpook Math. J., **25**(2) (1985), 123-126.

43. T. Noiri, *Properties of hyperconnected sets,* Acta Math. Hungar., **66** (1995), 147-154.

44. V. Popa, T. Noiri, *Weakly β-continuous functions,* An. Univ. Timişoara Ser. Mat. Inform., **32** (1994), 83-92.

45. I. L. Reilly, M. K. Vamanamurthy, *Connectedness and strong semi-continuity,* Čas. pěst. mat., **109** (1984), 261-265.

46. D. A. Rose, *Weak openness and almost openness,* Internat. J. Math & Math. Sci., 7(1) (1984), 35-40.

47. D. A. Rose, *Weak continuity and almost continuity,* Internat. J. Math. & Math. Sci., 7(2) (1984), 311-318.

48. M. K. Singal, Asha Rani Singal, *Almost-continuous mappings,* Yokohama Math. J., **16** (1968), 63-73.

49. T. Thompson, *Characterizations of irreducible spaces,* Kyungpook Math. J., **21** (2) (1981), 191-194.

50. A. Wilansky, *Topics in functional analysis,* Lecture Notes in Mathematics, vol. 45, Springer-Verlag 1967.

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