Jour. Pure Math., Vol. 31, 2014, pp. 29-35

A NOTE ON IDEALS OF $C(X)$

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ABSTRACT : For a topological space X, let $\mathcal{P}$ be an ideal of closed sub-sets of X and $C_{\mathcal{P}}(X)$ be the ideal of $C(X)$ of all functions f such that the support of f lies in $\mathcal{P}$. In this paper, we investigate the ideals of $C(X)$ which are of the form $C_{\mathcal{P}}(X)$ for some ideal $\mathcal{P}$ of closed sub-sets of X. We characterize P-spaces and almost P-spaces in terms of the ideals of the form $C_{\mathcal{P}}(X)$. Examples and counterexamples are given.

Key words : $C_{\mathcal{P}}(X)$, $C_K(X)$, P-space, almost P-space, F-space.

AMS subjclass [2010] Primary : 54C05, Secondary : 54G10, 54G05.

1. INTRODUCTION

Throughout, X will stand for a completely regular Hausdorff topological space, $C(X)$ denotes the ring of all real-valued continuous functions on X. For an $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ stands for the zero-set of f and $cl_X(X - Z(f))$ stands for the support of f. Let $\mathcal{P}$ be a family of closed subsets of X satisfying the following two conditions : (i) If $A, B \in \mathcal{P}$ then $A \cup B \in \mathcal{P}$. (ii) If $A \in \mathcal{P}$ and $B \subseteq A$ with B closed in X then $B \in \mathcal{P}$ i.e. $\mathcal{P}$ is an ideal of closed sets in X. In 2010, we initiated the ring $C_{\mathcal{P}}(X)$ for each ideal $\mathcal{P}$ of closed subsets of X as $C_{\mathcal{P}}(X) = \{f \in C(X) : cl_X(X - Z(f)) \in \mathcal{P}\}$, [1]. It is clear that $C_{\mathcal{P}}(X)$ is a z-ideal (possibly improper) of $C(X)$, an ideal I of $C(X)$ is called a z-ideal if $f \in I$, $Z(f) = Z(g)$ and $g \in C(X)$ imply that $g \in I$. It is also clear that if $\mathcal{P}$ denotes the family of all compact subsets of X then $C_{\mathcal{P}}(X)$ coincides with $C_K(X)$ where $C_K(X) = \{f \in C(X) : cl_X(X - Z(f))$ is compact$\}$. Again if $\mathcal{P}$ denotes the family of all closed subsets of X then $C_{\mathcal{P}}(X)$ coincides with $C(X)$.

Lemma 1.1. $C_{\mathcal{P}}(X) = C(X)$ if and only if $X \in \mathcal{P}$.

Proof. In fact, $C_{\mathcal{P}}(X) = C(X)$ if and only if $C_{\mathcal{P}}(X)$ contains units of $C(X)$ if and only if $X \in \mathcal{P}$.

Notations 1.2. (1) We denote the set of all ideals of closed sets in X by $\Omega(X)$ and the family of all ideals of $C(X)$ which are of the form $C_{\mathcal{P}}(X)$ for some $\mathcal{P} \in \Omega(X)$ by $\mathcal{J}_{\Omega(X)}$.

It is clear that $\Omega(X)$ is closed with respect to arbitrary intersection.

(2) Suppose that I is an ideal of $C(X)$. Consider the family of all members of $\Omega(X)$ containing $\{cl_X(X - Z(f)) : f \in I\}$. This family is nonempty since it contains the ideal of all closed subsets of X. Since $\Omega(X)$ is closed with respect to arbitary interesection, there exists a smallest member of $\Omega(X)$ containing $\{cl_X(X - Z(f)) : f \in I\}$ which we denote by $\mathcal{P}(I)$.

Note 1.3. It is obvious that $\{cl_X(X - Z(f)) : f \in I\}$ is closed with respect to finite union. Thus if $A \in \mathcal{P}(I)$ then $A \subseteq cl_X(X - Z(f))$ for some $f \in I$.

We now prove the following two lemmas.

Lemma 1.4. $\mathcal{J}_{\Omega(X)}$ is closed with respect to arbitrary intersection.

Proof. Let $\mathcal{J}_0 \subseteq \mathcal{J}_{\Omega(X)}$. Then $\mathcal{J}_0 = \{C_{\mathcal{P}}(X) : \mathcal{P} \in \Omega_0\}$ for some $\Omega_0 \subseteq \Omega(X)$. Since $\Omega(X)$ is closed with respect to arbitrary intersection, $\mathcal{P}_0 = \cap\{\mathcal{P} : \mathcal{P} \in \Omega_0\} \in \Omega(X)$. Also $\cap\mathcal{J}_0 = \cap\{C_{\mathcal{P}}(X) : \mathcal{P} \in \Omega_0\} = C_{\mathcal{P}_0}(X)$. Since $\mathcal{P}_0 \in \Omega(X)$, we see that $\cap\mathcal{J}_0 \in \mathcal{J}_{\Omega(X)}$.

Lemma 1.5. Suppose I is an ideal of $C(X)$. Then $C_{\mathcal{P}(I)}(X)$ is the smallest member of $\mathcal{J}_{\Omega(X)}$ containing I.

Proof. Obviously, $I \subseteq C_{\mathcal{P}(I)}(X)$. Let $I \subseteq C_{\mathcal{P}}(X)$ where $\mathcal{P} \in \Omega(X)$. Then $\{cl_X(X - Z(f)) : f \in I\} \subseteq \mathcal{P}$. Also $\mathcal{P}(I)$ is the smallest member of $\Omega(X)$ containing $\{cl_X(X - Z(f)) : f \in I\}$. Hence $\mathcal{P}(I) \subseteq \mathcal{P}$ and therefore $C_{\mathcal{P}(I)}(X) \subseteq C_{\mathcal{P}}(X)$.

Corollary 1.6. Suppose I is an ideal of $C(X)$. Then $I \in \mathcal{J}_{\Omega(X)}$ if and only if $I = C_{\mathcal{P}(I)}(X)$.

As usual βX denotes the Stone-Cech compactification of X. The maximal ideals of $C(X)$ are given by the family $\{M^p : p \in \beta X\}$ where $M^p = \{f \in C(X) : p \in cl_{\beta X}Z(f)\}$. Also for each $p \in \beta X$, the set $O^p = \{f \in C(X) : cl_{\beta X}Z(f)$ is a neighbourhood of $p\}$ is an ideal of $C(X)$. It is to be noted that M^p and O^p are z-ideals for all $p \in \beta X$. If $p \in X$ then we write M_p and O_p instead of M^p and O^p respectively. Thus $M_p = \{f \in C(X) : p \in Z(f)\}$ and $O_p = \{f \in C(X) : Z(f)$ is a neighbourhood of $p\}$. We now state the following theorem from Gillman-Jerison text, [7.12, [3]].

Theorem 1.7. Let $p \in \beta X$. Then $f \in O^p$ if and only if there is a neighbourhood V of p in βX such that $V \cap X \subseteq Z(f)$.

We now prove the following theorem.

Theorem 1.8. For each $p \in \beta X$, $O^p \in \mathcal{J}_{\Omega(X)}$.

Proof. Let $p \in \beta X$ and $O^p = I$. Consider the ideal $C_{\mathcal{P}(I)}(X)$ and suppose $f \in C_{\mathcal{P}(I)}(X)$. Then $cl_X(X - Z(f)) \in \mathcal{P}(I)$ and so $cl_X(X - Z(f)) \subseteq cl_X(X - Z(g))$ for some $g \in I = O^p$ (Note 1.3). Hence $int_X Z(g) \subseteq int_X Z(f)$. Now since $g \in O^p$, by Theorem 1.7, we find an open set V *in* βX containing p such that $V \cap X \subseteq Z(g)$. Since $V \cap X$ is open in X, we have $V \cap X \subseteq int_X Z(g)$. Thus $V \cap X \subseteq int_X Z(f)$. So by Theorem 1.7, $f \in O^p = I$. Hence $C_{\mathcal{P}(I)}(X) \subseteq I$ and therefore $I = C_{\mathcal{P}(I)}(X)$. Thus $O^p = I \in \mathcal{J}_{\Omega(x)}$.

A subset Z of a space X is called a zero-set if $Z = Z(f)$ for some $f \in C(X)$. A subset A of a space X is called regular closed if $A = cl_X int_X A$. Let us now prove the following theorems.

Theorem 1.9. Let $A \subseteq X$ be such that $cl_X A$ and $cl_X int_X A$ are both zero-sets in X. Then the following conditions are equivalent.

(1) $\cap_{p \in A} M_p = \cap_{p \in \mathrm{int}_X cl_X A} O_p$.

(2) $\cap_{p \in A} M_p \in \mathcal{J}_{\Omega(x)}$.

(3) $cl_X A$ is regular closed.

Proof. (1) $\Rightarrow$ (2) Follows from Theorem 1.8 and Lemma 1.4.

(2) $\Rightarrow$ (3) : Put $cl_X A = B$ and choose two functions $f, g \in C(X)$ such that $Z(f) = B$ and $Z(g) = cl_X int_X B$. Then $int_X Z(f) = int_X B \subseteq int_X Z(g)$. Also $int_X Z(g) \subseteq int_X Z(f)$ since $Z(g) \subseteq Z(f)$. Therefore $int_X Z(f) = int_X Z(g)$ and hence $cl_X(X - Z(f)) = cl_X(X - Z(g))$. Now $f \in \cap_{p \in A} M_p$ since $A \subseteq Z(f)$. Also by assumption, $\cap_{p \in A} M_p = C_{\mathcal{P}}(X)$ for some $\mathcal{P} \in \Omega(X)$. Thus $f \in C_{\mathcal{P}}(X)$. So $cl_X(X - Z(f)) \in \mathcal{P}$ and hence $g \in C_{\mathcal{P}}(X)$ since $cl_X(X - Z(g)) = cl_X(X - Z(f))$. Now it is obvious that $\cap_{p \in A} M_p = \cap_{p \in cl_X A} M_p$. So $\cap_{p \in A} M_p = \cap_{p \in B} M_p$. Thus $g \in \cap_{p \in B} M_p$. Consequently, $B \cap Z(g)$ and therefore $B \subseteq cl_X int_X B$. Hence $B = cl_X int_X B$. Thus $B = cl_X A$ is regular closed.

(3) $\Rightarrow$ (1) : Obviously, $\cap_{p\in A} M_p \subseteq \cap_{p\in int_X cl_X A} O_p$. Suppose now that $f \in \cap_{p\in int_X cl_X A} O_p$. Hence $int_X cl_X A \subseteq Z(f)$ and thus $cl_X int_X cl_X A \subseteq Z(f)$. Since $cl_X A$ is regular closed, $cl_X A \subseteq Z(f)$. Thus $A \subseteq Z(f)$ and consequently, $f \in \cap_{p\in A} M_p$. Hence $\cap_{p\in A} M_p = \cap_{p\in int_X cl_X A} O_p$.

Corollary 1.10. Suppose $\{p\}$ is a zero-set in a space X. Then $M_p \in \mathcal{J}_{\Omega(X)}$ if and only if p is an isolated point of X.

Proof. We note that $cl_X\{p\} = \{p\}$. Also $cl_X int_X cl_X\{p\} = \{p\}$ or $\emptyset$ according as p is an isolated point or not. Thus if $\{p\}$ is a zero-set then $cl_X\{p\}$ and $cl_X int_X cl_X\{p\}$ both are zero-sets. Taking $A = \{p\}$, from Theorem 1.9 we now can say that $M_p \in \mathcal{J}_{\Omega(X)}$ if and only if $cl_X\{p\}$ is regulr closed i.e. if and only if $cl_X\{p\} = cl_X int_X cl_X\{p\}$ i.e. if and only if p is an isolated point of X.

Example 1.11. The Corollary 1.10 becomes false if $\{p\}$ is not a zero-set in X. Take $X = [0, \omega_1]$, where ω_1 is the first uncountable ordinal. Each $f \in C(X)$ is eventually constant on a tail $[\alpha, \omega_1]$ for some $\alpha < \omega_1$, hence $\{\omega_1\}$ is not a zero-set in X. But "$\{\omega_1\}$ is a P-point of X", [5O.1, [3]] and thus $M_{\omega_1} = O_{\omega_1}$. Consequently, by Theorem 1.8, $M_{\omega 1} \in \mathcal{J}_{\Omega(X)}$ although, ω_1 is not an isolated point of X.

Example 1.12. Suppose $A = (0, 1)$, $B = [0, 1]$, $C = [0, 1] \cap \mathbb{Q}$ and $D = [0, 1] \cup \{2\}$. Then $cl_{\mathbf{R}}A$, $cl_{\mathbf{R}}B$, $cl_{\mathbf{R}}C$ are regular closed but $cl_{\mathbf{R}}D$ is not. So from Theorem 1.9 it follows that $\cap_{p\in A} M_p$, $\cap_{p\in B} M_p$, $\cap_{p\in C} M_p \in \mathcal{J}_{\Omega(\mathbf{R})}$ but $\cap_{p\in D} M_p \notin \mathcal{J}_{\Omega(\mathbf{R})}$. Again if $p \in \mathbf{R}$ then $M_p \notin \mathcal{J}_{\Omega(\mathbf{R})}$ as follows from Theorem 1.10.

2. *P*-SPACE, ALMOST *P*-SPACE, *F*-SPACE

A space X is called a P-space if $M_p = O_p$ for each $p \in X$. Equivalently, X is a P-space if every zero-set in X is open. In 2010, we characterized P-spaces in the following theorem, [Theorem 5.4, [1]].

Theorem 2.1. A space X is a P-space if and only if every ideal of $C(X)$ is of the form $C_{\mathcal{P}}(X)$ for some suitable family $\mathcal{P}$ of subsets of X with $\mathcal{P} \in \Omega(X)$.

From Theorem 2.1 we can say that if X is a P-space then each prime ideal of $C(X)$ is in $\mathcal{J}_{\Omega(X)}$. Interestingly, the converse is also true. In fact, if X is not a P-space then $M_p \neq O_p$ for some $p \in X$. Hence there exists a prime ideal P in $C(X)$ containing O_p which is not a z-ideal, [4I-5, 6, [3]]. Thus $P \notin \mathcal{J}_{\Omega(X)}$ since each member of $\mathcal{J}_{\Omega(X)}$ is a z-deal. Hence we have the following theorem.

Theorem 2.2. For a space X, the following are equivalent.

(1) X is a P-space.

(2) Every ideal of $C(X)$ is in $\mathcal{J}_{\Omega(X)}$.

(3) Every prime ideal of $C(X)$ is in $\mathcal{J}_{\Omega(X)}$.

A collection $\mathcal{F}$ of zero-sets in a space X is called a z-filter on X if (1) $\emptyset \notin \mathcal{F}$, (2) $\mathcal{F}$ is closed with respect to finite interesection and (3), $Z \in \mathcal{F}$ and Z_1 is a zero-set in X with $Z_1 \supseteq Z$ imply that $Z_1 \in \mathcal{F}$, [2.2, [3]]. Recall that if I is an ideal of $C(X)$ then the family $Z[I] = \{Z(f) : f \in I\}$ is a z-filter on X, [2.3 (a), [3]]. A space X is called an almost P-space if the interior of every nonempty zero-set in X is nonempty. It is well-known that a space X is an almost P-space if and only if every zero-set in X is regular closed, [Proposition 1.1, [4]]. In the following theorem we characterize almost P-spaces.

Theorem 2.3. For a space X, the following are equivalent.

(1) X is an almost P-space.

(2) $I \in \mathcal{J}_{\Omega(X)}$ for each z-ideal I of $C(X)$.

(3) $M^p \in \mathcal{J}_{\Omega(X)}$ for each $p \in \beta X$.

(4) $M_p \in \mathcal{J}_{\Omega(X)}$ for each $p \in X$.

Proof. (1) $\Rightarrow$ (2) : Let I be a z-ideal of $C(X)$. Suppose $f \in C_{\mathcal{P}(I)}(X)$. Then $cl_X(X - Z(f)) \in \mathcal{P}(I)$ and therefore $cl_X(X - Z(f)) \subseteq cl_X(X - Z(g))$ for some $g \in I$. Hence $int_X Z(g) \subseteq int_X Z(f)$ and so $cl_X int_X Z(g) \subseteq cl_X int_X Z(f)$. Since X is an almost P-space, every zero-set in X is regular closed and thus $Z(g) \subseteq Z(f)$. Also $g \in I$ shows that $Z(g) \in Z[I]$. Since $Z[I]$ is z-filter on X we now have $Z(f) \in Z[I]$. Thus $Z(f) = Z(h)$ for some $h \in I$. Hence $f \in I$ since I is a z-ideal. Thus $C_{\mathcal{P}(I)}(X) \subseteq I$ and so $I = C_{\mathcal{P}(I)}(X)$.

(2) ⇒ (3) : Trivial since every maximal ideal in $C(X)$ is a z-ideal.

(3) ⇒ (4) : Trivial.

(4) ⇒ (1) : Suppose (1) is false. Then there is a nonempty zero-set, say Z in X such that $int_X Z = \emptyset$. Choose $p \in Z$ and suppose $Z = Z(f)$ where $f \in C(X)$. Then $f \in M_p$. Thus $cl_X(X - Z(f)) \in \mathcal{P}(I)$ where $I = M_p$. Now $cl_X(X - Z(f)) = X - int_X Z = X$ since $int_X Z = \emptyset$. Hence $X \in \mathcal{P}(I)$. Therefore $C_{\mathcal{P}(I)}(X) = C(X)$. thus $M_p = I \subsetneq C_{\mathcal{P}(I)}(X)$. From Corollary 1.6 it now follows that $M_p = I \notin \mathcal{J}_{\Omega(X)}$. Hence (4) is false.

We note that if $M^p \in \mathcal{J}_{\Omega(X)}$ for each $p \in \beta X - X$ then X need not be an almost P-space. Consider the following example.

Example 2.4. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Suppose $\Sigma = \mathbb{N} \cup \{\sigma\}$ where $\sigma \notin \mathbb{N}$. Define a topology on Σ as follows : all points on $\mathbb{N}$ are isolated and the neighbourhoods of σ are the sets $U \cup \{\sigma\}$ for $U \in \mathcal{U}$, [4M, [3]]. In Σ, the set $\{\sigma\}$ is a zero-set. Also $int_\Sigma\{\sigma\} = \emptyset$. So Σ is not an almost P-space. Now choose $p \in \beta\Sigma - \Sigma$ and suppose $M^p = I$. If $M^p \subsetneq C_{\mathcal{P}(I)}(\Sigma)$ then $C_{\mathcal{P}(I)}(\Sigma) = C(\Sigma)$ since M^p is maximal. Therefore $int_\Sigma Z(f) = \emptyset$ for some $f \in I = M^p$. Now $f \in M^p$ shows that $p \in cl_{\beta\Sigma} Z(f)$ and therefore $Z(f)$ is not compact. So $Z(f)$ contains points of $\mathbb{N}$. Since all point of $\mathbb{N}$ are isolated, it now follows that $int_\Sigma Z(f) \neq \emptyset$, a contradiction. Hence $M^p = C_{\mathcal{P}(I)}(\Sigma)$. Thus $M^p \in \mathcal{J}_{\Omega(\Sigma)}$ for each $p \in \beta\Sigma - \Sigma$.

An abstract ring R is called an F-ring if each finitely generated ideal in R is principal. A space X is called an F-space if $C(X)$ is an F-ring. Equivalently, X is an F-space if and only if for each $f \in C(X)$ there exists $k \in C(X)$ such that $f = k|f|$, [14.25, [3]]. In 2014, we characterized F-spaces in terms of the ideals $C_{\mathcal{P}}(X)$, [Theorem 2.1, [2]]. We now prove the following theorem.

Theorem 2.5. Consider the following conditions for a space X.

(1) Every finitely generated ideal in $C(X)$ is in $\mathcal{J}_{\Omega(X)}$.

(2) Every principal ideal in $C(X)$ is in $\mathcal{J}_{\Omega(X)}$.

(3) X is an F-space.

Then (1) and (2) are equivalent and each of them implies (3).

Proof. (1) $\Rightarrow$ (2) : Trivial.

(2) $\Rightarrow$ (3) : Choose $f \in C(X)$. By (2), $(|f|) = C_{\mathcal{P}}(X)$ for some $\mathcal{P} \in \Omega(X)$. Hence $(|f|)$ is a z-ideal. Now $Z(|f|) = Z(f)$ shows that $f \in (|f|)$. Hence there exists $k \in C(X)$ such that $f = k|f|$. Hence X is an F-space.

(2) $\Rightarrow$ (1) : If (2) is true then from (2) $\Rightarrow$ (3) above we see that X is an F-space. Thus every finitely generated ideal in $C(X)$ is principal. Hence the proof follows.

It is to be noted that in Theorem 2.5, (3) need not imply (2) or (1). We consider the following example.

Example 2.6. Consider the space Σ described in Example 2.4. It is an F-space, [4M-8, [3]]. Since $\{\sigma\}$ is a zero-set in Σ, we can select an $f \in C(\Sigma)$ such that $Z(f) = \{\sigma\}$. If possible now let $I = (f) \in \mathcal{J}_{\Omega(\Sigma)}$. Then $I = C_{\mathcal{P}}(\Sigma)$ for some $\mathcal{P} \in \Omega(\Sigma)$. Therefore $cl_{\Sigma}(\Sigma - Z(f)) \in \mathcal{P}$. Hence $cl_{\Sigma}(\Sigma - \{\sigma\}) \in \mathcal{P}$. Since σ is not isolated in Σ, $cl_{\Sigma}(\Sigma - \{\sigma\}) = \Sigma$. Thus $\Sigma \in \mathcal{P}$ and consequently, $I = C_{\mathcal{P}}(\Sigma) = C(\Sigma)$, which is not possible since f is not a unit in $C(\Sigma)$. Therefore $I = (f) \notin \mathcal{J}_{\Omega(\Sigma)}$.

Acknowledgement. the author is thankful to Dr. Bikram Banerjee, Assistant Professor of Mathematics, Ranaghat College, India and also to the learned referee for their valuable suggestions towards the improvement of the paper.

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