Journal of Pure Mathematics; Vol. 32(2023), 18 – 29

Journal of Pure Mathematics

Some curves in the framework of three dimensional f-Kenmotsu manifolds

Srimayee Samui

Department of Mathematics, Vidyasagar College, 39, Sankar Ghosh Lane, Kolkata-700006, West Bengal, India, e-mail : srimayee.samui@gmail.com

Pradip Majhi *

Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata-700019, West Bengal, India, e-mail : mpradipmajhi@gmail.com or pmpm@caluniv.ac.in

Abhijit Biswas

Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kolkata-700019, West Bengal, India, e-mail : abhibiswas1991@gmail.com

Abstract

We obtain the differential equations for characterizing Frenet curves, Legendre curves and magnetic curves in three dimensional f-Kenmotsu manifolds. Also we prove that under certain assumptions a Frenet curve whose curvature and torsion are given is Legendre curve.

AMS Subject Classification : Primary 53D15; Secondary 35Q51.

Key words : Frenet curve, Legendre curve, Magnetic curve, Curvature, torsion, f-Kenmotsu manifolds.

*Corresponding author.

1 Introduction

A nice notion of classical differential geometry of curves is that of curves of constant slope, also called cylindrical helix. This is a curve in the Euclidean space E^3 for which the tangent vector field has a constant angle with a fixed direction called the axis. The second name corresponds to the fact that there exist a cylinder on which the curves moves in such a way that it cuts each ruling at a constant angle. The classical characterization of these curves is the Bertrand-Lancret-de Saint Venant Theorem([4]): The curve γ in E^3 is of constant slope if and only if the ratio of the torsion τ and the curvature κ is constant. More precisely, for a cylindrical helix we have the constant ratio $\frac{\cos\theta}{|\sin\theta|} = \frac{\tau}{\kappa}$ and then, inspired by the title of [4], in paper [2] the authors define the Lancret invariant as Lancret$(\gamma) = \frac{\cos\theta}{|\sin\theta|}$. By computing κ and τ in terms of θ we get the result above and therefore the expression of Lancret invariant in the 3-dimensional Euclidean geometry is:

$$Lancret(\gamma) = \frac{\tau}{\kappa}, \tag{1}$$

An interesting generalization of this class of curves is that slant curve in almost contact metric geometry. This concept was introduced in [8] with the constant angle θ between the tangent and the Reeb vector field. The particular case of $\theta = \frac{\pi}{2}(or\theta = \frac{3\pi}{2})$ is very important since we recover the Legendre curves of [2]

In [9], Cabrerizo, Fernandez and Gomez introduced a geometric approach to the study of magnetic fields on three dimensional Sasakian manifolds. A curve γ is called magnetic curve in 3-dimensional f-Kenmotsu manifolds if $\nabla_{\dot{\gamma}}\dot{\gamma} = \phi\dot{\gamma}$. A magnetic curve is the trajectory of magnetic fields. Geodesics on a manifold are curves which do not experience any kind of forces where the magnetic curves experience due to magnetic fields. If the magnetic field disappears, the magnetic curves become geodesics. In this way a magnetic curve is a generalization of a geodesic.

In the study of f-Kenmotsu manifolds, Legendre curves play a important role. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in the paper [2]. Belkhelfa et al [6] have investigated Legendre curve in Riemannian and Lorentzian manifolds and many others such as ([27], [28]).

Let M be a 3-dimensional Riemannian manifold. Let $\gamma : I \to M, I$ being an interval, be a curve in M which is parameterized by arc length, and let $\nabla_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to the Levi-Civita connection on M. It is said that γ is a Frenet curve if one of the following three cases holds:

- γ is of osculating order 1, i.e,$\nabla_{V_1}V_1 = 0$ (geodesic), $V_1 = \dot{\gamma}$. Here, . denotes differentiation with respect to the arc parameter.

- γ is of osculating order 2, i.e., there exist two orthonormal vector fields $V_1(= \dot{\gamma}), V_2$ and a non-negative functions κ (curvature) along γ such that $\nabla_{V_1}V_1 = \kappa V_2$, $\nabla_{V_1}V_2 = -\kappa V_1$.

- γ is of osculating order 3, i.e., there exist three orthonormal vectors $V_1(= \dot{\gamma}), V_2, V_3$

and two non-negative functions κ(curvature) and τ(torsion) along γ such that

$$\nabla_{V_1} V_1 = \kappa V_1, \tag{2}$$

$$\nabla_{V_1} V_2 = -\kappa V_1 + \tau V_3, \tag{3}$$

$$\nabla_{V_1} V_3 = -\tau V_2. \tag{4}$$

With respect to the Levi-Civita connection, a Frenet curve of osculating order 3 for which k is a positive constant and $\tau = 0$ is called a circle in M ; a Frenet curve of osculating order 3 is called a helix in M if κ and τ both are positive constants and the curve is called a generalized helix if $\frac{\kappa}{\tau}$ is a constant.

2 Preliminaries

Let M be an almost contact manifold i.e, M is a connected $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) [3]. As usually denote by Φ the fundamental 2-form of M, $\Phi(X,Y) = g(X, \phi Y), X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of differentiable vector fields on M.

- normal if the almost complex structure defined on the the product manifold $M \times \mathbb{R}$ is integrable (equivalently $[\phi,\phi] + 2d\eta \otimes \xi = 0$),
- almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$, For further use, we recall the following definitions [3], [10], [26]. The manifold M and its structure (ϕ, ξ, η, g) is said to be:
- cosymplectic if it is normal and almost cosymplectic(equivalently,$\nabla\phi = 0$, ∇ being covariant differentiation with respect to the Levi-Civita connection)

The manifold M is called locally conformal cosymplectic (respectively, almost cosymplectic) if M has an open covering U_t endowed with differentiable functions $\sigma_t : U_t \to \mathbb{R}$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \xi_t = e^{\sigma_t}\xi, \eta_t = e^{-\sigma_t}\eta, g_t = e^{-2\sigma_t} g \tag{5}$$

is cosymplectic (respectively, almost cosymplectic)
Olszak and Rosca [23] studied normal locally conformal almost cosymplectic manifold.They gave a geometric interpretation of f-Kenmotsu manifolds and studied some curvature properties. Among others they proved that a Ricci symmetric f-Kenmotsu manifold is an Einstein manifold.
By an f-Kenmotsu manifolds we mean an almost contact metric manifold which is normal and locally conformal almost cosymplectic.
Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1 \tag{6}$$

$$\phi\xi = 0, \eta\circ\phi = 0, \eta(X) = g(X,\xi) \tag{7}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{8}$$

for any vector fields $X, Y \in \chi(M)$ where I is the identity of the tangent bundle TM, ϕ is a tensor field of $(1,1)$-type, η is a 1-form, ξ is a vector field and g is a metric tensor field. We say that (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies [24]:

$$(\nabla_X \phi)(Y) = f(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{9}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha = constant \neq 0$, then the manifold is a α-Kenmotsu manifold [14]. 1-Kenmotsu manifold is a Kenmotsu manifold ([15], [25]). If $f = 0$, then the manifold is cosymplectic [14]. An f-Kenmotsu manifold is said to be regular if $f^2 + f' \neq 0$, where $f' = \xi f$.
For an f-Kenmotsu manifold from (2.2) it follows that

$$\nabla_X \xi = f(X - \eta(X)\xi) \tag{10}$$

The condition $df \wedge \eta = 0$ holds if dim $M \geq 5$. In general this does not hold if dim$M = 3$ [23].
In a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X,Y)Z \quad &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &-\frac{r}{2}g(Y,Z)X - g(X,Z)Y \end{aligned} \tag{11}$$

In a 3-dimensional f-Kenmotsu manifold, we have [23]

$$\begin{aligned} R(X,Y)Z \quad &= (\frac{r}{2} + 2f^2 + 2f')(g(Y,Z)X - g(X,Z)Y) \\ &-(\frac{r}{2} + 3f^2 + 3f')\{\eta(X)(g(Y,Z)\xi - g(\xi,Z)Y) \\ &+\eta(Y)(g(\xi,Z)X - g(X,Z)\xi)\} \end{aligned} \tag{12}$$

$$S(X,Y) = (\frac{r}{2} + 2f^2 + 2f')g(Y,Z)X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y) \tag{13}$$

where r is a scaler curvature of M and $f' = \xi(f)$.
From (9), we obtain

$$R(X,Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y] \tag{14}$$

and (10) yields

$$S(X,\xi) = -(f^2 + f')\eta(X) \tag{15}$$

3 Characterization of curves in three dimensional f-Kenmotsu manifold

In this section first we characterize Frenet curves and then Legendre curves and magnetic curves.

Let $\gamma : I \to M$ parameterized by the arc length parameter, be a non-geodesic Frenet curve in three dimensional contact metric manifold. Differentiating (2) with respect to V_1, we have

$$\begin{aligned} \nabla^2_{V_1} V_1 &= \nabla_{V_1}(\kappa V_2) \\ &= \dot{\kappa} V_2 - \kappa^2 V_1 + \kappa\tau V_3. \end{aligned} \tag{16}$$

Differentiating (16) with respect to V_1, we obtain

$$\begin{aligned} \nabla^3_{V_1} V_1 &= \nabla_{V_1}(\dot{\kappa} V_2 - \kappa^2 V_1 + \kappa\tau V_3 \\ &= -2\kappa\dot{\kappa} V_1 - \kappa^2 \nabla_{V_1} V_1 + \dot{\kappa}\nabla_{V_1} V_2 + \ddot{\kappa} V_2 \\ &\quad + \dot{\kappa}\tau V_3 + \kappa\dot{\tau} V_3 + \kappa\tau\nabla_{V_1} V_3. \end{aligned} \tag{17}$$

V_3 is taken from the equation (16) and use V_2 from the equation (2), we get

$$\begin{aligned} V_3 &= \frac{1}{\kappa\tau}\nabla^2_{V_1} V_1 + \frac{\kappa^2}{\kappa\tau} V_1 - \frac{\dot{\kappa}}{\kappa\tau} V_2 \\ &= \frac{1}{\kappa\tau}\nabla^2_{V_1} V_1 - \frac{\dot{\kappa}}{\kappa\tau}\nabla_{V_1} V_1 + \frac{\kappa}{\tau} V_1. \end{aligned} \tag{18}$$

Differentiating (17) with respect to V_1, we have

$$\begin{aligned} \nabla_{V_1} V_3 &= \frac{1}{\kappa\tau}\nabla^3_{V_1} V_1 - \frac{\kappa\dot{\tau} + \dot{\kappa}\tau}{(\kappa\tau)^2}\nabla^2_{V_1} V_1 \\ &\quad - \frac{\kappa^2\tau\ddot{\kappa} - 2\kappa\dot{\kappa}^2\tau - \kappa^2\dot{\kappa}\dot{\tau}}{(\kappa^2\tau)^2}\nabla_{V_1} V_1 \\ &\quad - \frac{\dot{\kappa}}{\kappa\tau}\nabla^2_{V_1} V_1 + \frac{\tau\dot{\kappa} - \kappa\dot{\tau}}{\tau^2} V_1 + \frac{\kappa}{\tau}\nabla_{V_1} V_1. \end{aligned} \tag{19}$$

Then using the Frenet equation, we get

$$\begin{aligned} -\tau V_2 &= \frac{1}{\kappa\tau}\nabla^3_{V_1} V_1 - \frac{\kappa\dot{\tau} + 2\dot{\kappa}\tau}{(\kappa\tau)^2}\nabla^2_{V_1} V_1 \\ &\quad + \left(\frac{\kappa}{\tau} - \frac{\kappa^2\tau\ddot{\kappa} - 2\kappa\dot{\kappa}^2\tau - \kappa^2\dot{\kappa}\dot{\tau}}{(\kappa^2\tau)^2}\right)\nabla_{V_1} V_1 \\ &\quad + \frac{\tau\dot{\kappa} - \kappa\dot{\tau}}{\tau^2} V_1. \end{aligned} \tag{20}$$

$$\begin{aligned} -\frac{\tau}{\kappa}\nabla_{V_1} V_1 &= \frac{1}{\kappa\tau}\nabla^3_{V_1} V_1 - \frac{\kappa\dot{\tau} + 2\dot{\kappa}\tau}{(\kappa\tau)^2}\nabla^2_{V_1} V_1 \\ &\quad + \left(\frac{\kappa}{\tau} - \frac{\kappa^2\tau\ddot{\kappa} - 2\kappa\dot{\kappa}^2\tau - \kappa^2\dot{\kappa}\dot{\tau}}{(\kappa^2\tau)^2}\right)\nabla_{V_1} V_1 \\ &\quad + \frac{\tau\dot{\kappa} - \kappa\dot{\tau}}{\tau^2} V_1. \end{aligned} \tag{21}$$

Therefore we have

$$\nabla^3_{V_1}V_1 - \frac{\kappa\dot\tau + 2\dot\kappa\tau}{\kappa\tau}\nabla^2_{V_1}V_1 + (\kappa^2+\tau^2 - \frac{\kappa^2\tau\ddot\kappa - 2\kappa\dot\kappa^2\tau - \kappa^2\dot\kappa\dot\tau}{\kappa^3\tau})\nabla_{V_1}V_1 + \frac{\kappa}{\tau}(\tau\dot\kappa - \kappa\dot\tau)V_1 = 0. \quad (22)$$

Hence we can state the following:

Theorem 3.1. *If γ be a unit-speed non-geodesic Frenet curve in contact metric manifold, then it satisfies the following equation*

$$\nabla^3_{V_1}V_1 + \lambda_1\nabla^2_{V_1}V_1 + \lambda_2\nabla_{V_1}V_1 + \lambda_3 V_1 = 0 \quad (23)$$

where

$$\lambda_1 = -\frac{\kappa\dot\tau + 2\dot\kappa\tau}{\kappa\tau},$$

$$\lambda_2 = \kappa^2+\tau^2 - \frac{\kappa^2\tau\ddot\kappa - 2\kappa\dot\kappa^2\tau - \kappa^2\dot\kappa\dot\tau}{\kappa^3\tau},$$

$$\lambda_3 = \frac{\kappa}{\tau}(\tau\dot\kappa - \kappa\dot\tau).$$

Let γ be a non-geodesic legendre curve on three dimensional f-Kenmotsu manifold M^3. Then for the Frenet frame with components (V_1, V_2, V_3) of the curve γ, the values of curvature κ and torsion τ are given by ([19])

$$\kappa = \sqrt{f^2+\delta^2}$$

and

$$\tau = \frac{f\dot\delta - \delta\dot f}{f^2+\delta^2}.$$

Differentiating κ and τ with respect to V_1, we get

$$\dot\kappa = \frac{f\dot f + \delta\dot\delta}{\sqrt{f^2+\delta^2}} = \frac{f\dot f + \delta\dot\delta}{\kappa},$$

$$\ddot\kappa = \frac{(f\ddot f + \delta\ddot\delta + \dot f^2 + \dot\delta^2) - \dot\kappa^2}{\kappa},$$

and

$$\dot\tau = \frac{(f\ddot\delta - \delta\ddot f) - 2\kappa\tau\dot\kappa}{\kappa^2}.$$

For Legendre curve γ, using the value of κ, τ, $\dot\kappa$, $\ddot\kappa$ and $\dot\tau$ in λ_1 λ_2 and λ_3 in the Theorem (3.1), we get

$$\lambda_1 = -\frac{f\ddot\delta - \delta\ddot f}{f\dot\delta - \delta\dot f}, \quad (24)$$

$$\lambda_2 = (f^2+\delta^2) + \frac{(f\dot\delta - \delta\dot f)^2}{(f^2+\delta^2)^2} + \frac{(f\dot f + \delta\dot\delta)^2}{(f^2+\delta^2)^2} + \frac{(f\dot f + \delta\dot\delta)(f\ddot\delta - \delta\ddot f)}{(f^2+\delta^2)(f\dot\delta - \delta\dot f)} - \frac{f\ddot f + \delta\ddot\delta + \dot f^2 + \dot\delta^2}{f^2+\delta^2} \quad (25)$$

and

$$\lambda_3 = 3(f\dot{f} + \delta\dot{\delta}) - \frac{(f^2+\delta^2)(f\ddot{\delta} - \delta\ddot{f})}{f\dot{\delta} - \delta\dot{f}}. \tag{26}$$

Therefore we have the following theorem

Theorem 3.2. *If γ be a unit-speed non-geodesic Legendre curve in three dimensional f-Kenmotsu manifold, then it satisfies the following*

$$\nabla^3_{V_1} V_1 + \lambda_1 \nabla^2_{V_1} V_1 + \lambda_2 \nabla_{V_1} V_1 + \lambda_3 V_1 = 0 \tag{27}$$

where λ_1, λ_2 and λ_3 are given by (24),(25) and (26).

Let γ be a magnetic curve on three dimensional f-Kenmotsu manifold M^3. Then for the Frenet frame with components (V_1, V_2, V_3) of the curve γ, the value of curvature κ and torsion τ are given by ([20])

$$\kappa = \sqrt{1-\sigma^2} \tag{28}$$

and

$$\tau = \sigma \tag{29}$$

where

$$\sigma = \eta(\dot{\gamma}) = g(\dot{\gamma}, \xi). \tag{30}$$

Differentiating (30) with respect to $V_1(=\dot{\gamma})$, we have

$$\dot{\sigma} = f(1-\sigma^2). \tag{31}$$

Differentiating κ two times and τ one times with respect to the V_1, we get

$$\dot{\kappa} = -f\sigma\sqrt{1-\sigma^2} = -f\sigma\kappa = -f\kappa\tau \tag{32}$$

$$\begin{aligned} \ddot{\kappa} &= (-\dot{f}\sigma - f^2 + 2f^2\sigma^2)\sqrt{1-\sigma^2} \\ &= -\dot{f}\kappa\tau - f\dot{\kappa}\tau - f\kappa\dot{\tau} \end{aligned} \tag{33}$$

$$\dot{\tau} = \dot{\sigma} = f(1-\sigma^2) = f\kappa^2 \tag{34}$$

Here we see that

$$\kappa^2 + \tau^2 = 1 \tag{35}$$

Using the value of κ, τ, $\dot{\kappa}$, $\ddot{\kappa}$ and $\dot{\tau}$ of the magnetic curve in λ_1 λ_2 and λ_3 in the theorem(3.1), we get

$$\lambda_1 = -f\frac{1-2\sigma^2}{\sigma}, \tag{36}$$

$$\lambda_2 = 1 + \dot{f}\sigma + f^2\sigma^2 \tag{37}$$

$$\lambda_3 = -f\frac{1-\sigma^2}{\sigma} \tag{38}$$

In view of the above equations we can state the following:

Theorem 3.3. *If γ be a unit-speed non-geodesic magnetic curve in three dimensional f-Kenmotsu manifold, then it satisfies the following*

$$\nabla^3_{V_1}V_1 + \lambda_1\nabla^2_{V_1}V_1 + \lambda_2\nabla_{V_1}V_1 + \lambda_3 V_1 = 0 \tag{39}$$

where λ_1, λ_2 and λ_3 are given by (36),(37) and (38).

For Legendre magnetic curve, $\kappa = 1$ and $\tau = 0$. Using value of κ and τ in (17), we get

$$\nabla^3_{V_1}V_1 + \nabla_{V_1}V_1 = 0. \tag{40}$$

Therefore we get the following theorem

Theorem 3.4. *A unit-speed Legendre magnetic curve γ in three dimensional f-Kenmotsu manifold satisfies the following equation*

$$\nabla^3_{V_1}V_1 + \lambda_1\nabla^2_{V_1}V_1 + \lambda_2\nabla_{V_1}V_1 + \lambda_3 V_1 = 0 \tag{41}$$

with $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 0$.

Now for the magnetic slant curve but not Legendre curve, κ and τ are constants. Then we have also the following theorem

Theorem 3.5. *A unit-speed magnetic slant curve γ in three dimensional f-Kenmotsu manifold satisfies the following*

$$\nabla^3_{V_1}V_1 + \lambda_1\nabla^2_{V_1}V_1 + \lambda_2\nabla_{V_1}V_1 + \lambda_3 V_1 = 0 \tag{42}$$

with $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 0$.

4 When a Frenet curve will be a Legendre curve in a three dimensional f-Kenmotsu manifold?

Theorem 4.1. *Let M be a three dimensional f-Kenmotsu manifold. Let $\gamma : I \to M$ be a Frenet curve in M, with curvature $\kappa > 0$ and torsion $\tau = |\frac{f\dot{\delta} - \dot{f}\delta}{\kappa^2}|$. Let $\sigma = \eta(\dot{\gamma})$ and $\sigma(t_0) = \sigma(\dot{t_0}) = \sigma(\ddot{t_0}) = 0$ at a certain point $t_0 \in I$, then γ is a Legendre curve.*

Proof. Let M be a three dimensional f-Kenmotsu manifold. Let $\gamma : I \to M$ be a Frenet curve in M, with curvature $\kappa > 0$ and torsion $\tau = |\frac{f\dot{\delta} - \dot{f}\delta}{\kappa^2}|$.
$\sigma = \eta(\dot{\gamma})$.
Let $\sigma(t_0) = \sigma(\dot{t_0}) = \sigma(\ddot{t_0}) = 0$ at a certain point $t_0 \in I$. We will show that there exists a neighbourhood $I_0 \subset I$ of t_0 on which $\sigma = 0$.
Firstly we choose a neighbourhood $I_1 \subset I$ of t_0, on which $|\sigma| < 1$. $\dot{\gamma}$ is not colinear with ξ at every point I_1. The vector fields

$$\dot{\gamma}, \frac{\phi\dot{\gamma}}{\sqrt{1-\sigma^2}}, \frac{\xi - \sigma\dot{\gamma}}{\sqrt{1-\sigma^2}}$$

From an orthonormal frame along γ, and consequently we have

$$\nabla_{\dot{\gamma}} E_1 = \nabla_{\dot{\gamma}} \dot{\gamma} = a \frac{\phi \dot{\gamma}}{\sqrt{1-\sigma^2}} + b \frac{\xi - \sigma \dot{\gamma}}{\sqrt{1-\sigma^2}} \tag{43}$$

by choosing certain functions a and $b = m \frac{\dot{\gamma}}{\sqrt{1-\sigma^2}}$, m being an arbitrary constant on I_1. Therefore

$$\kappa^2 = a^2 + b^2. \tag{44}$$

Now differentiating $\sigma = \eta(\dot{\gamma})$ we have

$$\dot{\sigma} = \dot{\gamma}(g(\xi, \dot{\gamma})) = g(\nabla_{\dot{\gamma}}) + g(\xi, \nabla_{\dot{\gamma}\dot{\gamma}}). \tag{45}$$

Applying (10) and (43) we have

$$\dot{\sigma} = f(1-\sigma^2) + b\sqrt{1-\sigma^2}. \tag{46}$$

From Frenet formula we obtain

$$E_2 = \frac{1}{\kappa} \nabla_{\dot{\gamma}} E_1 = \frac{a}{\kappa\sqrt{1-\sigma^2}} \phi \dot{\gamma} + \frac{b}{\kappa\sqrt{(1-\sigma^2)}} (\xi - \sigma \dot{\gamma}) \tag{47}$$

Let us consider $a_1 = \frac{a}{\kappa\sqrt{1-\sigma^2}}$ and $b_1 = \frac{b}{\kappa\sqrt{(1-\sigma^2)}}$. Thus

$$E_2 = a_1 \phi \dot{\gamma} + b_1 (\xi - \sigma \dot{\gamma}). \tag{48}$$

Taking differentiation of a_1 and b_1 we have

$$\dot{a_1} = \frac{b(\dot{a}b - a\dot{b})}{\kappa^3\sqrt{1-\sigma^2}} + \frac{a\sigma f}{\kappa\sqrt{(1-\sigma^2)}} + \frac{ab\sigma}{\kappa(1-\sigma^2)}$$

$$\dot{b_1} = \frac{a(a\dot{b} - \dot{a}b)}{\kappa^3\sqrt{1-\sigma^2}} + \frac{b\sigma f}{\kappa\sqrt{(1-\sigma^2)}} + \frac{b^2\sigma}{\kappa(1-\sigma^2)}$$

Now differentiating (48) we obtain

$$\nabla_{\dot{\gamma}} E_2 = \dot{a_1} \phi \dot{\gamma} + a_1((\nabla_{\dot{\gamma}\phi})\dot{\gamma} + \phi \nabla_{\dot{\gamma}} \dot{\gamma}) + \dot{b_1}(\xi - \sigma \dot{\gamma}) + b_1(\nabla_{\dot{\gamma}} \xi - \dot{\sigma} \dot{\gamma} - \sigma \nabla_{\dot{\gamma}} \dot{\gamma}). \tag{49}$$

Putting the value of $\dot{a_1}$ and $\dot{b_1}$ and using (44),(9) and (10) and after certain long calculations we have

$$\nabla_{\dot{\gamma}} E_2 = a_2 \frac{\phi \dot{\gamma}}{\sqrt{1-\sigma^2}} + b_2 \frac{(\xi - \sigma \dot{\gamma})}{\sqrt{1-\sigma^2}} - \kappa \dot{\gamma} \tag{50}$$

where $a_2 = -\frac{b}{\kappa} c$ and $b_2 = \frac{a}{\kappa} c$ with

$$c = \frac{a\dot{b} - b\dot{a}}{\kappa^3} - \frac{a\sigma}{\kappa\sqrt{1-\sigma^2}}. \tag{51}$$

Thus

$$a_2^2 + b_2^2 = c^2. \tag{52}$$

Again from Frenet formula we have

$$\tau E_3 = \nabla_{\dot{\gamma}} E_2 + \kappa E_2.$$

Using (50) in the above equation we obtain

$$\tau E_3 = a_2 \frac{\phi\dot{\gamma}}{\sqrt{1-\sigma^2}} + b_2 \frac{(\xi - \sigma\dot{\gamma})}{\sqrt{1-\sigma^2}} - \kappa\dot{\gamma} + \kappa\dot{\gamma}. \tag{53}$$

Therefore

$$\tau = \sqrt{a_2^2 + b_2^2}. \tag{54}$$

From (52) and (54) we can conclude that

$$\tau = |c|.$$

According to our assumption $\sigma(t_0) = \dot{\sigma}(t_0) = \ddot{\sigma}(t_0)$ and depending our choice of b we can state $b(t_0) = \dot{b}(t_0) = 0$.
From (46) we have $f(t_0) = \dot{f}(t_0) = 0$. $a(t_0) \neq 0$, since $b(t_0) = 0$ and $\kappa(t_0) > 0$.
As $\tau = |c|$ and $c = \frac{a\dot{b}-b\dot{a}}{\kappa^3} - \frac{a\sigma}{\kappa\sqrt{1-\sigma^2}}$, $\tau = |\frac{a\dot{b}-b\dot{a}}{\kappa^3} - \frac{a\sigma}{\kappa\sqrt{1-\sigma^2}}| = 0$ at t_0. Thus there exists a neighbourhood I_0 of t_0 such that $\tau = |\frac{a\dot{b}-b\dot{a}}{\kappa^3} - \frac{a\sigma}{\kappa\sqrt{1-\sigma^2}}| = 0$ which implies that

$$\frac{a\dot{b} - b\dot{a}}{\kappa^3} - \frac{a\sigma}{\kappa\sqrt{1-\sigma^2}} = 0 \tag{55}$$

Again

$$b = m\frac{\dot{\sigma}}{\sqrt{1-\sigma^2}}, \dot{b} = m\frac{1}{\sqrt{1-\sigma^2}}(\ddot{\sigma} + \frac{\sigma\dot{\sigma}^2}{1-\sigma^2}).$$

Putting these values in equation (55) we obtain

$$a(\ddot{\sigma} + \frac{2m\sigma\dot{\sigma}^2}{1-\sigma^2}) + \frac{a^3\sigma}{m} - \dot{\sigma}\dot{a} = 0. \tag{56}$$

From [7] using initial conditions $\sigma(t_0) = \dot{\sigma}(t_0) = 0$ and $a \neq 0$ we claim that $\sigma = 0$ on I_0. Now in similar way of [29] we can prove that $\sigma = 0$ on the whole I which implies that σ is a Legendre curve. This completes the proof.

Acknowledgement : The authors are very much thankful to the anonymous referee for some valuable comments

References

[1] J. Arroya, O. Garay, J. J. Mencia, *When is a periodic function the curvature of a closed plane curve ?*, The American Mathematical Monthly, 115(2008), 405-414.

[2] Ch. Baikoussis and D. E. Blair, *On Legendre curves in contact 3-manifolds*, Geom. Dedicata 49 (no. 2)(1994), 135-142 .

[3] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics Vol.509, Springer-Verlag, Berlin-New York, (1976).

[4] M. Barros, *General helices and a theorem of Lancret*, Proc. Amer. Math. Soc.125(no 5)(1997), 1503-1509.

[5] B. Y. Chen, S. Ishikawa, *Biharmonic surfaces in pseudo Euclidean spaces*, Mem. Fac. Sci.Kyushu Univ. Ser. A., 45(1991), 323-347.

[6] M. Belkhalfa, I. E. Hirica, R. Rosaca and L. Verstraelen, *On Legendre curves in Riemannian and Sasakian spaces*, Soochow J. Math., 28(2002), 81-91.

[7] E. A. Coddington, *An Introduction to Ordinary Differential Equations*, New York, Dover, 1989.

[8] J. T. Cho, J. I. Inoguchi and J. E. Lee, *On slant curves in Sasakian 3-manifolds*, Bull. Austral. Math. Soc. 74(no. 3)(2006), 359-367.

[9] J. L. Cabrerizo, M. Fernandez, J. S. Gomez, *The contact magnetic flow in 3D Sasakian manifolds*, J. Phys. A.Math., Theor., 42(2009), 195201(10pp).

[10] S. I. Goldberg and K.Yano, *Integrability of almost cosymplectic structures*, Pacific J. Math., 31(1969), 373-382 .

[11] M. Hasan Sahid, *CR-submanifolds of trans-Sasakian manifolds*, Indian J. of Pure and Appl. Math., 22(1991), 1007-1012.

[12] J. I. Inoguchi, J. E. Lee, *Biminimal curves in two dimensional space forms*, Commun Korean Math Soc., 27(2012), 771-780.

[13] J. I. Inoguchi, *Submanifolds with harmonic mean curvature vector field in contact 3-manifold*, Cooloq. Math. 100(no. 2)(2004),163-167.

[14] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J., 4(1981), 1-27.

[15] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24(1972), 93-103.

[16] H. Kocayigit and H. H. Hacisalihoglu, 1-*Type curves and biharmonic curves in euclidean 3-space*, Int. Elec. J. Geom, 4(2011), 97-101.

[17] J. Langer, A. Singer, *The total squared curvature of closed curves*, J. Differential Geom., 20(1984), 1-22.

[18] E. Loubeau, S. Montaldo, *Biminimal immersions*, Proc. Edinb. Math. Soc., 51(2008), 421-437.

[19] P. Majhi, A. Biswas, *Some special curves in three dimensional f-Kenmotsu manifolds*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., 27(2)(2020), 83-96.

[20] P. Majhi, A. Biswas, *Certain curves associated with f-Kenmotsu manifolds*, J. Dynam. Sys. Geom. Theor.,18(1)(2020),39-51.

[21] P. Majhi, *Almost ricci solution and gradient almost ricci solution on 3-dimensional f-Kenmotsu manifolds*, Kyungpook Math. J. 57(2017), 309-318.

[22] S. Montaldo, C. Oniciuc, *A short survey on biharmonic maps between Riemannian manifolds*, Rev. Un. Mat.Argentina, 47(2006), 1-22.

[23] Z. Olszak and R. Rosca, *Normal locally conformal almost cosymplectic manifolds*, Publ. Math. Debrecen, 39(1991), 315-323.

[24] Z. Olszak, *Locally conformal almost cosymplectic manifolds*, Colloq. Math., 57(1989), 73-87.

[25] G. Pitis, A remark on Kenmotsu manifolds, Bul. Univ. Brasov Ser. C, 30(1988), 31-32.

[26] S. Sasaki and Y.Hatakeyama, *On differentiable manifolds with certain structures which are closely related to almost contact structures II*, Tohoku Math. j., 13(1961), 281-294.

[27] A. Sarkar and A. Sil, *Curves on some classes of Kenmotsu manifolds*. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 59 (2016), 101-112.

[28] A. Sarkar and A. Mondal, *Certain curves on some classes of three-dimensional almost contact metric manifolds*. Rev. Un. Mat. Argentina 58 (2017), no. 1, 107-125.

[29] J. Welyczko, *On Legendre curves in* 3*-dimensional normal almost contact metric manifolds*, Soochow Journal of Mathematics, 33(2007), 929-937.