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On Weak Compactness of Variable Exponent Spaces

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Abstract

This work shows some refined necessary and sufficient conditions placed on the subsets of variable exponent Lebesgue spaces to satisfy the axiom of weak compactness. We also present some results in connection with conditions for all separable variable exponent spaces to be weakly Banach-saks. That is, some results on the Banach-Saks property in variable exponent spaces are given.

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Key words : Variable exponent Lebesgue spaces, Weak compactness, Equi-integrability, Weak Banach-Saks, Cesàro convergence.

1 Introduction

The extension of the Riesz-Kolmogorov theorem in classical L^p-spaces $(1 \leq p < \infty)$ has been recently done to the notion of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ (where the constant p is replaced with the variable $p(\cdot)$) by Gòrka and Macios [8], Gòrka and Bandaliyev [7] and Dong et al. [4]. Useful versions of the above mentioned theorem with underlying measure spaces like Euclidean spaces, locally compact group or metric measure spaces were authored by the aforementioned researchers. Over the past two decades, the notion of variable exponent spaces are being used in several areas of harmonic analysis and differential equations and applications (see [9], [3]). The notion of variable exponent spaces belong to a generalized class of non-symmetric (rearrangement variant) Orlicz spaces (see [10], [13]). In a paper ([11]) by F. L. Hernàndez, C. Ruiz and M. Sanchiz, weakly compact sets are derived in a non-reflexive variable exponent spaces $L^{p(\cdot)}(\Omega)$. The notion has been widely studied for symmetric (rearrangement invariant) function spaces. For Orlicz spaces

$L^{\varphi}(\Omega)$ with the Δ_2-condition, useful weak compactness criteria were given by Andô in [2]. In ([11]), the extension of the Andô weak compactness characterization in Orlicz spaces was carried out in variable exponent $L^{p(\cdot)}(\Omega)$ setting and, also equi-integrable subsets in the variable exponent spaces $L^{p(\cdot)}(\Omega)$ were studied by obtaining a De la Vallèe Poussin type theorem ([12]) in $L^{p(\cdot)}(\Omega)$. They apply the De la Vallèe classical result by obtaining the criteria for when the inclusion between two variable exponent spaces $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ are weakly compact or L-weakly compact operators meaning that the unit ball $B_{L^{q(\cdot)}}$ is equi-integrable in $L^{p(\cdot)}(\Omega)$. It also turns out, in their work, that closed exponent functions $p(\cdot)$ and $q(\cdot)$, the inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ can be L-weakly compact. As an application, the authors in ([11]) obtained the weak compactness criteria which are useful in the study of the weak Banach-Saks property in $L^{p(\cdot)}(\Omega)$ spaces (that is, every weakly convergent sequence in $L^{p(\cdot)}(\Omega)$ contains a subsequence which is Cesàro convergent).
One of our goals in this paper is to redefine the modular function $\rho_{p(\cdot)}\Big(\frac{f}{r}\Big)$ and also likely introduce a new norm which is an inverse norm equivalent to the associated Luxemburg type norm to the modular function. We also redefine the essential range of the exponent function $p(\cdot)$ in terms of the ε-ball. Also, equi-integrable subsets in $L^{p(\cdot)}(\Omega)$ are studied and, we provide equivalent statement for the De la Vallèe Poussin type theorem ([5]) in $L^{p(\cdot)}(\Omega)$.

2 Preliminaries

Let $(\Omega, \sum, \mu)$ be a finite separable non-atomic measurable space and $L^0(\Omega)$ be the space of all real measurable functions. Given a μ-measurable function $p : \Omega \longrightarrow [1, \infty)$ or $p : \Omega \longrightarrow \mathbb{R}^+$, the variable exponent Lebesgue space (or Nakano space), denoted by $L^{p(\cdot)}(\Omega)$, is the set of all measurable scalar function classes $f \in L^0(\Omega)$ such that the modular function $\rho_{p(\cdot)}\Big(\frac{f}{r}\Big)$ is finite for some $r > 0$, where

$$\rho_{p(\cdot)}\Big(f\Big) = \int_{\Omega} \Big|f(t)\Big|^{p(t)} d\mu(t) < \infty$$

That is,

$$L^{p(\cdot)}(\Omega) = \Big\{f \in L^0(\Omega) : \rho_{p(\cdot)}\Big(\frac{f}{r}\Big) < \infty \quad \text{for some} \quad r > 0\Big\}$$

where the modular function can be redefined as

$$\rho_{p(\cdot)}\Big(\frac{f}{r}\Big) = \int_{\Omega} \Big|\frac{f(t)}{r}\Big|^{p(t)} d\mu(t) = |r|^{-p(t)} \int_{\Omega} \Big|f(t)\Big|^{p(t)} d\mu(t) < \infty$$

The associated Luxemburg type norm is defined as

$$\| f \|_{p(\cdot)} = \inf \Big\{r > 0 : \rho_{p(\cdot)}\Big(\frac{f}{r}\Big) \leq 1\Big\}$$

The new norm equivalent to the above norm is given as

$$\| f \|'_{p(\cdot)} = \sup \Big\{\frac{1}{r} > 0 : \rho_{p(\cdot)}\Big(fr\Big) \leq 1\Big\} = \frac{1}{\inf \Big\{r > 0 : \rho_{p(\cdot)}\Big(\frac{f}{r}\Big) \leq 1\Big\}}$$

That is, $\| \cdot \|'_{p(\cdot)} = \frac{1}{\| \cdot \|_{p(\cdot)}}$ since $\| f \|_{p(\cdot)} \neq 0$. Th variable exponent space $L^{p(\cdot)}(\Omega)$ with the norm $\| \cdot \|'_{p(\cdot)}$, is Banach lattice.
Consider the following definitions.

Definition 2.1. Let $p : \Omega \longrightarrow \mathbb{R}_+$ be an exponent function defined on Ω. Then the essential infimum and supremum are respectively given as

$$p^+ := \text{essinf}\Big\{p(t) : t \in \Omega\Big\} \quad \text{and} \quad p^- := \text{esssup}\Big\{p(t) : t \in \Omega\Big\}$$

Let $p^+_{|_A}(\cdot)$ and $p^-_{|_A}(\cdot)$ denote the supremum and infimum of the function $p(\cdot)$ over a measurable subset $A \subset \Omega$. The topological dual of the space $L^{p(\cdot)}(\Omega)$ for $p^+ < \infty$, is the variable exponent space $L^{p^*(\cdot)}(\Omega)$, where $p^*(\cdot)$ is the conjugate function of $p(\cdot)$ such that $\frac{1}{p(\cdot)} + \frac{1}{p^*(\cdot)} = 1$ almost everywhere $t \in \Omega$.

Definition 2.2. A variable exponent space $L^{p(\cdot)}(\Omega)$ is separable if and only if the essential supremum of $p : \Omega \longrightarrow \mathbb{R}^+$ is finite. That is, $p^+ < \infty$ or similarly, if and only if $L^{p(\cdot)}(\Omega)$ contains no isomorphic copy of l^∞.
Throughout this work, we deal with the separable $L^{p(\cdot)}(\Omega)$. $L^{p(\cdot)}(\Omega)$ is reflexive if and only if $1 < p^- \leq p^+ < \infty$. This is similar to $L^{p(\cdot)}(\Omega)$ being uniformly convex. That is, a varibale exponent space $L^{p(\cdot)}(\Omega)$ is uniformly convex if and only if $1 < p^- \leq p^+ < \infty$.
Recall that $\| \cdot \|_{p(\cdot)} = 1$ for $p^+ < \infty$ if and only if the modular $\rho_{p(\cdot)}(f) = 1$. Also, every sequence $(\xi_n) \subset L^{p(\cdot)}(\Omega)$ satisfies the condition:

$$\lim_{n\to\infty} \| \xi_n \|_{p(\cdot)} 0 = \quad \text{if and only if} \quad \lim_{n\to\infty} \rho_{p(\cdot)}(\xi_n) = 0$$

The essential range of the exponent function p is defined as

$$\mathfrak{R}_{p(\cdot)} = \Big\{q \in \mathbb{R}^+ : \forall \quad \varepsilon > 0, \mu\Big(p^{-1}(|\xi - q|) < \varepsilon\Big) > 0\Big\}$$

where $\mu\Big(p^{-1}(|\xi - q|) < \varepsilon\Big) > 0$ can also be constructed as $\mu\Big(p^{-1}(B_\varepsilon(q))\Big) > 0$, and $B_\varepsilon(q) := \Big\{\xi : |\xi - q| < \varepsilon\Big\}$. Let $B_{L^{p(\cdot)}}$ denotes the closed unit ball of $L^{p(\cdot)}(\Omega)$. The essential range, $\mathfrak{R}_{p(\cdot)}$, is a closed subset of $\mathbb{R}^+$ and $\mathfrak{R}_{p(\cdot)}$ is compact if $p(\cdot)$ is essentially bounded. Both the values p^- and p^+ are embedded in the essential range of $p(\cdot)$. For every $q \in \mathfrak{R}_{p(\cdot)}$ there exists a suitable sequence of disjoint measurable subsets (A_k) such that the normalized sequence, defined by

$$g_k = \frac{\chi_{A_k}}{\Big(\mu(A_K)\Big)^{\frac{p^*(\cdot)-1}{p^*(\cdot)}}}$$

is equivalent to the canonical form of l^q, where $\mu(A_k) \neq 0$,

$$\chi_{A_k}(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in A_k \\ 0 & \text{if} \quad \xi \notin A_k \end{cases}$$

Remark 2.3. Variable exponent spaces are a special class of Musielak-Orlicz spaces. An Orlicz function $\varphi : \mathbb{R}^+_0 \longrightarrow [0, \infty]$ is a convex increasing function if $\varphi(0) = 0$, $\lim_{x\to 0^+} \varphi(x) = 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.
A function $\Psi : \Omega \times [0, \infty) \longrightarrow [0, \infty]$ is a Musielak-Orlicz function if $\Psi(\xi, \cdot)$ is an Orlicz function $\forall \quad \xi \in \Omega$ and $\xi \mapsto \Psi(\xi, x)$ is measurable for all $x \geq 0$.

Definition 2.4. Let $\Psi(\xi, x)$ be a Musielak-Orlicz function. The Orlicz space $L^{\psi}(\Omega)$ is defined as the set of all measurable scalar functions on Ω such that $\rho_{\Psi}\left(\frac{f}{r}\right)$ is finite for some $r > 0$, where $\rho_{\Psi}(\cdot)$ is the modular function defined by

$$\rho_{\Psi}(f) = \int_{\Omega} \Psi\Big(\xi, |f(\xi)|\Big) d\mu(\xi) < \infty$$

In some cases if $\Psi(\xi, x) = x^{p(\xi)}$, then the Orlicz space L^{Ψ} is reduced to the variable exponent space $L^{p(\cdot)}(\Omega)$ and, $\Psi(\xi, x) = \varphi(x) \quad \forall \quad \xi \in \Omega$, then we get the Orlicz space $L^{\varphi}(\Omega)$.

Definition 2.5. Let X be a Banach function space and $S \subset X(\Omega)$ be a bounded subset of $X(\Omega)$. Then S is equi-integrable if

$$\lim_{\mu(A)\to 0} \sup_{f\in S} \int_{\Omega} \Big|f\chi_A(\xi)\Big|^{p(\xi)} d\mu(\xi) = \lim_{\mu(A)\to 0} \sup_{f\in S} \parallel f\chi_A \parallel_X = 0 \quad \text{provided} \quad X(\Omega) = L^{p(\cdot)}(\Omega)$$

In other way, in classical L^p-spaces, equi-integrability has special role in the notion of $L^{p(\cdot)}(\Omega)$ spaces which is defined as

$$\lim_{\mu(A)\to 0} \sup_{f\in S} \int_{\Omega} \Big|f\chi_A(\xi)\Big|^{p} d\mu = \lim_{\mu(A)\to 0} \sup_{f\in S} \parallel f\chi_A \parallel_X = 0 \quad \text{provided} \quad p(\cdot) = p$$

Some of the important roles of equi-integrability of $X(\Omega)$ can be found in Riesz-Kolomogorov compactness type theorem in $L^{p(\cdot)}(\Omega)$. Also, the boundedness of equi-integrable subsets in $L^1(\Omega)$ in some Orlicz spaces $L^{\varphi}(\Omega)$ is found in the classical De la Vallèe Poussin's result ([4]). The extension of this result to $L^{p(\cdot)}(\Omega)$ spaces and the equivalent statement of $L^{p^{\cdot}}(\Omega)$ equi-integrability was done in [11] as presented as the first result in the next section.

3 Some Equivalent Results On Equi-Integrability

Proposition 3.1. *Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $S \subset L^{p(\cdot)}(\Omega)$ bounded. Then S is equi-integrable if and only if*

$$\lim_{x\to\infty} \sup_{f\in S} \int_{|f|>x} \Big|f(t)\Big|^{p(t)} d\mu = 0$$

In the next result, we present the equivalence of Proposition 3.1

Proposition 3.2. *Let $L^{p(\cdot)}(\Omega)$ be a variable exponent space with $p^+ < \infty$. Then a bounded subset $S \subset L^{p(\cdot)}(\Omega)$ is uniformly integrable if and only if*

$$\lim_{\xi\to\infty} \sup_{f\in S} \int_{\xi\notin A} \Big|f\chi_A(\xi)\Big|^{p(\xi)} d\mu(\xi) = 0 \quad or \quad \sup_{f\in S} \parallel f\chi_A(\xi) \parallel_{p(\cdot)} \leq \varepsilon.$$

Proof. Let $p^+ < \infty$ and S be bounded and uniformly integrable. We show that

$$\lim_{\xi\to\infty} \sup_{f\in S} \int_{\xi\notin A} \Big|f\chi_A(\xi)\Big|^{p(\xi)} d\mu(\xi) = 0 \quad \Longrightarrow \sup_{f\in S} \parallel f\chi_A(\xi) \parallel_{p(\cdot)} \leq \varepsilon$$

Since $f \in S$ is bounded, then let $\sup_{f\in S} \parallel f\chi_A(\xi) \parallel_{p(\cdot)} \leq M < \infty$.

Define the sets $A_p^{\xi} = \Big(\text{supp}(f)\Big)^c = \Big\{t \in \Omega : |f(t)| > \xi\Big\}$, where $\text{supp}(f) = \overline{\Big\{\xi \in \Omega : f(\xi) \neq 0\Big\}}$.

Now, we show that $\lim_{\xi\to\infty} \sup_{f\in S} \mu(A_p^\xi) = 0$

Whence,

$$\begin{aligned}\sup_{f\in S} \mu(A_p^\xi) \le \sup_{f\in S} \mu\Big(\text{supp}(f)\Big)^c \le \frac{\alpha}{\xi} \sup_{f\in S} \parallel f\chi_{A_p^\xi} \parallel_{p(\cdot)} \le \frac{1}{\xi} \sup_{f\in S} \parallel f\chi_{A_p^\xi} \parallel_{p(\cdot)} \\ \le \sup_{f\in S} \frac{1}{\xi}\Big(1+\mu(\Omega)\Big) \parallel f\chi_{A_p^\xi} \parallel_{p(\cdot)} \le \frac{M}{\xi}\Big(1+\mu(\Omega)\Big) < \varepsilon\end{aligned}$$

provided $\xi > 0$ and $\alpha > 0$.

On the other hand, let $\varepsilon > 0$ then there exists $\xi > 1$ such that $\sup_{f\in S} \parallel f\chi_{A_p^\xi} \parallel_{p(\cdot)} \le \frac{\varepsilon}{2}$. Then for every measurable subset A with $\Big(\mu(A)\Big)^{\frac{1}{p^+}} < \frac{\varepsilon}{2\xi}$, we have

$$\begin{aligned}\sup_{f\in S} \parallel f\chi_A \parallel_{p(\cdot)} &\le \sup_{f\in S} \left(\left\| f\chi_{A\cap \text{supp}(f)^c} \right\|_{p(\cdot)} + \left\| f\chi_{A\cap \text{supp}(f)} \right\|_{p(\cdot)} \right) \\ &\le \sup_{f\in S} \left(\left\| f\chi_{A\cap \text{supp}(f)^c} \right\|_{p(\cdot)} + \Big(\mu(A)\Big)^{\frac{1}{p^+}} \xi \right) \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Thus, $\lim_{\xi\to\infty} \sup_{f\in S} \parallel f\chi_A(\xi) \parallel_{p(\cdot)} = 0$. □

Theorem 3.3. *A bounded $S \subset L^{p(\cdot)}(\Omega)$ is equi-integrable if and only if there exists an Orlicz function with $\lim_{x\to 0^+} \varphi(0) = 0$ for $p^+ < \infty$ such that*

$$\sup_{f\in S} \parallel \varphi(f)(\xi) \parallel_{p(\cdot)} \le \varepsilon.$$

Proof. Assume that S is equi-integrable. Consider a convex increasing function (ξ_n) such that, for $\xi_n > 2\xi_{n-1}$ for $n = 2, 3, \cdots$

$$\sup_{f\in S} \left\| f\chi_{-\xi_{n-1}>f>\xi_{n-1}} \right\|_{p(\cdot)} \le \frac{1}{(n-1)^2}$$

Defining a function $\varphi(\xi)$ by

$$\varphi(\xi) = \sum_{n=2}^{\infty} (\xi - \xi_{n-1}), \quad \text{for} \quad \xi \ge 0$$

Moreover, $\lim_{\xi\to 0^+} \varphi(\xi) = 0$. For the interval $[\xi_{n-1}, \xi_n)$, we have the partial sum

$$\varphi(\xi) = \sum_{\alpha=2}^{n} (\xi - \xi_{\alpha-1}) = n\xi \sum_{\alpha=2}^{n} \xi_{\alpha-1} \ge n\xi - 2\xi_{n-1}$$

Hence, for all $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that

$$\parallel \varphi(f) \parallel_{p(\cdot)} \le \sum_{n=2}^{\infty} \parallel f\chi_{-\xi_{n-1}>f>\xi_{n-1}} \parallel_{p(\cdot)} \le \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \frac{\pi^2}{6}$$

Therefore, $\sup_{f\in S} \parallel \varphi(f) \parallel_{p(\cdot)} \le \sup \frac{\pi^2}{6} \le \varepsilon$.

Conversely, given $\varepsilon > 0$, we assume that $\sup_{f\in S} \parallel \varphi(f) \parallel_{p(\cdot)} \le \varepsilon$ by hypothesis. We show that the bounded $f \in S$ is equi-integrable. For every $\xi \ge \xi_\varepsilon$, we have $\xi \le \varepsilon\varphi(\xi)$. Then we have

$$\left\| f\chi_{|f|>\xi_{\varepsilon-1}} \right\|_{p(\cdot)} \le \varepsilon \left\| \varphi(f)\chi_{|f|>\xi_{\varepsilon-1}} \right\|_{p(\cdot)} \le \varepsilon \sup_{f\in S} \left\| \varphi(f) \right\|_{p(\cdot)}$$

Hence the proof. □

Definition 3.4. Let $p, q : \Omega \longrightarrow \mathbb{R}^+$ be exponent functions such that $p(\cdot) \leq q(\cdot)$. The inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is said to be L-weakly compact if the unit ball $B_{L^{q(\cdot)}}$ is equi-integrable in $L^{p(\cdot)}(\Omega)$.

Let S be the unit ball $B_{L^{q(\cdot)}}$ in the Theorem 3.3 to get following result:

Proposition 3.5. *Let $p(\cdot) \leq q(\cdot) \leq r(\cdot)$ be exponent functions. The inclusion $L^{r(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is L-weakly. That is, there exists an Orlicz function φ with $\lim_{\xi \to 0^+} \varphi(\xi) = 0$ such that $L^{q(\cdot)}(\Omega) \subset L^{\Psi}(\Omega)$ where the Musielak-Orlicz function Ψ is given by $\Psi(\xi, x) = \Big(\varphi(x)\Big)^{p(\xi)}$.*

Consider the following result by Hernàndez, Ruiz, Sanchiz (2021).

Proposition 3.6. *Let $p(\cdot) \leq q(\cdot)$ be exponent functions. If $ess \inf \Big(q(x) - p(x)\Big) = \delta > 0$, then the inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is said to be L-weakly.*

Consider the equivalent statement of the Proposition 3.6.

Proposition 3.7. *Let $p, q : \Omega \longrightarrow \mathbb{R}^+$ be exponent functions such that $p(\cdot) \leq q(\cdot)$. If $ess \inf \Big(p(\xi) - q(\xi)\Big) = \delta > 0$, then the inclusion $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is said to be L-weakly.*

Proof. We want to show that the limit of the modular function tends to zero. That is,

$$\lim_{\mu(A) \to 0} \sup_{||f|| \leq 1} \rho_{p(\cdot)} f \chi_A = 0$$

Let $r(\xi) = \frac{p(\xi)}{q(\xi)} \leq 1$. Denote the exponent function with conjugate function by $r^*(\xi) = \frac{p(\xi)}{p(\xi) - q(\xi)}$ for $\xi \in \Omega$. It applies that $(r^*)^+ \leq \dfrac{p^+}{\delta} < 0 < \infty$. Using Hölder's inequality, we have

$$\rho_{q(\cdot)}(f\chi_A) = \int_\Omega |f|^{q(\cdot)} \chi_A d\mu \leq 4 \parallel f^{q(\cdot)} \parallel_{r(\cdot)} \parallel \chi_A \parallel_{r^*(\cdot)}$$

Also,

$$\rho_{r(\cdot)}\Big(f^{q(\cdot)}\Big) = \int_\Omega \Big|f\Big|^{p(\xi)} d\mu \leq \parallel f \parallel_{p(\cdot)}^{p^+} \leq 1$$

When $\parallel f^{q(\cdot)} \parallel_{r(\cdot)} \leq 1$, we have

$$\lim_{\mu(A) \to 0} \sup_{||f|| \leq 1} \rho_{p(\cdot)} \Big(f \chi_A\Big) \leq \lim_{\mu(A) \to 0} 4 \parallel f^{q(\cdot)} \parallel_{r(\cdot)} = 0$$

□

We present a weaker condition.

Proposition 3.8. *Let $p(\cdot) \leq q(\cdot)$ be exponent functions in $\Omega = [0, \alpha]$ with $q^+ < \infty$ and $(p - q)(\cdot)$ increasing or non-decreasing. Assume that $|(\alpha - x)^{p-q}| < \varepsilon$, then*

(i) $\lim_{x \to \alpha} (\alpha - x)^{(p-q)(x)} = 0$, *and*

(ii) There exists a sequence (x_n) defined by $x_n = \frac{x_{n-1}+\alpha}{2} \quad \forall \quad n \in \mathbb{N}$, and $x_0 \in [0, \alpha)$ satisfying that

$$\sum_{n=1}^{\infty} \frac{1}{x_n - x_{n-1}} \int_{x_{n-1}}^{x_n} \Big(x_n - x_{n-1}\Big)^{\frac{(p-q)(t)}{p(t)}} dt < \infty$$

Then, the inclusion $L^{q(\cdot)}\Big([0, \alpha]\Big) \subset L^{p(\cdot)}\Big([0, \alpha]\Big)$ is L-weakly compact.

Proof. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{n_0 \in \mathbb{N}} \frac{1}{x_n - x_{n-1}} \int_{x_{n-1}}^{x_n} \Big(x_n - x_{n-1}\Big)^{\frac{(p-q)(t)}{p(t)}} dt < \frac{\varepsilon}{4}$$

and

$$\Big(x_n - x_{n-1}\Big)^{\frac{(p-q)(t)}{p(t)}} \leq \Big(\alpha - x_n\Big)^{\frac{(p-q)(t)}{N}} < \frac{\varepsilon}{4} \tag{1}$$

for $\dfrac{(p-q)}{p} \leq \dfrac{(p-q)}{N}$ where $N = p^+ = \operatorname*{ess\,sup}_{x \in \Omega} \quad \forall \quad x \in [x_{n-1}, x_n), \quad n \geq n_0$.
Assume that $r = (p-q)(x_{n_0}) < 0$. Let $f \in B_{L^{q(\cdot)}}$ and any measurable set E with $\mu(E) \leq \left(\frac{\varepsilon}{8}\right)^{\frac{N}{r}+\alpha}$. We define the following sets

$$E_1 = \left\{x \in [0, x_{n_0}) \cap E : |f(x)| \leq \left(\frac{8}{\varepsilon}\right)^{\frac{1}{r}}\right\}, \quad E_2 = \left\{x \in [0, x_{n_0}) \cap E : |f(x)| > \left(\frac{8}{\varepsilon}\right)^{\frac{1}{r}}\right\}$$

where $\mathrm{supp}(f) = E_1$ and $\Big(\mathrm{supp}(f)\Big)^c = E_2$.
Then for $f \in B_{L^{q(\cdot)}}$ and $\mu(E) \leq \left(\dfrac{\varepsilon}{8}\right)^{\frac{N}{r}+\alpha}$, we have

$$\begin{aligned}
\int_{[0,x_{n_0}0 \cap E} |f|^{p(t)} dt &= \int_{E_1} |f|^{p(t)} dt + \int_{E_2} |f|^{p(t)} dt \\
&\leq \left(\frac{\varepsilon}{8}\right)^{\frac{N}{r}} \mu(E) + \int_{E_2} |f|^{q(t)} |f|^{p(t)-q(t)} dt \\
&\leq \frac{\varepsilon}{8} + \int_{E_2} |f|^{q(t)} \frac{\varepsilon}{8} dt \leq \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{x_{n_0}}^{\alpha} |f|^{p(t)} \chi_E dt &= \sum_{n_0 \in \mathbb{N}} \int_{x_{n-1}}^{x_n} |f|^{p(t)} \chi_E dt \\
&= \sum_{n_0 \in \mathbb{N}} \int_{E_{n-1,1}} |f|^{p(t)} \chi_E dt + \int_{E_{n-1,2}}^{x_n} |f|^{p(t)} \chi_E dt
\end{aligned}$$

where

$$E_{n-1,1} = E \cap \left\{x \in [x_{n-1}, x_n) : |f(x)| \leq \frac{1}{\Big(x_n - x_{n-1}\Big)^{\frac{1}{q(x)}}}\right\}$$

$$E_{n-1,2} = E \cap \left\{x \in [x_{n-1}, x_n) : |f(x)| > \frac{1}{\Big(x_n - x_{n-1}\Big)^{\frac{1}{q(x)}}}\right\}$$

Using (1), we have

$$\sum_{n_0\in\mathbb{N}}\int_{E_{n-1,1}}|f|^{p(t)}dt \le \sum_{n_0\in\mathbb{N}}\int_{x_{n-1}}^{x_n}\frac{1}{\Big(x_n-x_{n-1}\Big)^{\frac{p(t)}{q(t)}}}dt$$

$$\le \sum_{n_0\in\mathbb{N}}\int_{x_{n-1}}^{x_n}\frac{1}{\Big(x_n-x_{n-1}\Big)}\Big(x_n-x_{n-1}\Big)^{\frac{(p-q)(t)}{p(t)}}dt < \frac{\varepsilon}{4}$$

Also, for $f\in B_{L^{q(\cdot)}}$, we have

$$\sum_{n_0\in\mathbb{N}}\int_{E_{n-1,2}}|f|^{p(t)}dt \le \sum_{n_0\in\mathbb{N}}\int_{x_{n-1}}^{x_n}|f|^{q(t)}|f|^{(p-q)(t)}dt$$

$$\le \sum_{n_0\in\mathbb{N}}\int_{x_{n-1}}^{x_n}|f|^{q(t)}\Big(x_n-x_{n-1}\Big)^{\frac{(p-q)(t)}{p(t)}}dt$$

$$\le \sum_{n_0\in\mathbb{N}}\int_{x_{n-1}}^{x_n}|f|^{q(t)}\frac{\varepsilon}{4}dt \le \frac{\varepsilon}{4}$$

Hence the result. □

Consider the equivalence of Theorem 4.2 in [11] about relatively weakly compactness of subset $L^{p(\cdot)}(\Omega)$.

Theorem 3.9. *Let $L^{p(\cdot)}(\Omega)$ be with $p^+<\infty$. A subset $S\subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if S is norm bounded and, $\forall \quad g\in L^{p^*(\cdot)}(\Omega)$, we have*

$$\lim_{\mu(E)\to 0}\sup_{f\in S}\int_E |fg|d\mu = \lim_{\mu(E)\to 0}\inf_{f\in S}\int_E |fg|d\mu = 0$$

Proof. Suppose that S is relatively weakly compact. Then we show that S is norm bounded. To see this, let $\varepsilon>0$ be given, then we have $g\in L^{p^*(\cdot)}(\Omega)$, a sequence (E_n) with $\mu(E_n)\to 0$ and $(f_n)\in S$. Since S is relatively weakly compact, there exists a subsequence $(f_{n_k})\to f\in L^{p(\cdot)}(\Omega)$ as $k\to\infty$. Thus, $\forall \quad A\in\sum$,

$$\lim_{\mu(A)\to 0}\sup_{f_{n_k}\in S}\int_\Omega |f_{n_k}g\chi_A|d\mu = \lim_{\mu(A)\to 0}\inf_{f_{n_k}\in S}\int_\Omega |f_{n_k}g\chi_A|d\mu = 0$$

from the notion that

$$\int_\Omega f_{n_k}g_0\chi_A d\mu \xrightarrow{k\to\infty} \int_\Omega f g_0 d\mu$$

On the other hand, let S be norm bounded and $p^+,p^-<\infty$. Since S is norm bounded, there exists a sequence $(f_n)\subset S$ with $\parallel f_n \parallel_{p(\cdot)}\le M<\infty$. From Bolzano-Weierstrass theorem, there exists a Cauchy subsequence (f_{n_k}) such that for all $g\in L^{p^*(\cdot)}(\Omega)$,

$$\int_\Omega\Big(f_{n_k}-f_{n_l}\Big)gd\mu \longrightarrow 0 \quad \text{as} \quad k,l\to 0 \quad \text{or} \quad \Big\|\Big(f_{n_k}-f_{n_l}\Big)g\Big\|_{p(\cdot)}<\varepsilon \quad \forall \quad n_k,n_l>N(\varepsilon)$$

Let us denote $G_m=\Big\{t\in\Omega:|f(t)|\le m\Big\}$. Since $g\in L^1(\Omega)$, consider large natural m such that $\mu(G_m^c)\le\delta$. With dominated convergence theorem, a simple function g_s such that

$\| g_m - g_s \|_{p^*(\cdot)} \leq \dfrac{\varepsilon}{24M}$. Thus, for k, l large enough so that $\int_\Omega \Big|\Big(f_{n_k} - f_{n_l}\Big)g_s\Big|d\mu < \dfrac{\varepsilon}{3}$, we use Hölder's inequality to get

$$\begin{aligned}\Big|\int_\Omega \Big(f_{n_k} - f_{n_l}\Big)gd\mu\Big| &= \Big|\int_{G_m} \Big(f_{n_k} - f_{n_l}\Big)gd\mu + \int_{G_m^c} \Big(f_{n_k} - f_{n_l}\Big)gd\mu\Big| \\ &\leq \int_{G_m} \Big|\Big(f_{n_k} - f_{n_l}\Big)g\Big|d\mu + \int_{G_m^c} \Big|\Big(f_{n_k} - f_{n_l}\Big)g\Big|d\mu \\ &\leq \int_\Omega \Big|\Big(f_{n_k} - f_{n_l}\Big)g\Big|d\mu + \frac{\varepsilon}{3} \\ &\leq \int_\Omega \Big|\Big(f_{n_k} - f_{n_l}\Big)(g_m - g_s)\Big|d\mu + \int_\Omega \Big|\Big(f_{n_k} - f_{n_l}\Big)g_s\Big|d\mu + \frac{\varepsilon}{3} \\ &\leq \varepsilon\end{aligned}$$

Therefore, we have (f_{n_k}) is a weakly Cauchy sequence. So, (f_{n_k}) is weakly convergent to a function $f \in L^{p(\cdot)}(\Omega)$ and S is relatively weakly compact. □

Let $L^{p(\cdot)}(\Omega)$ be a variable exponent space with $\Omega_1 = p^{-1}\Big(\{1\}\Big)$. Consider the equivalent statement of the Theorem 4.3 in [11] with similar proof.

Theorem 3.10. *Let $L^{p(\cdot)}(\Omega)$ with $p^+ < \infty$ and $\mu(\Omega_1) = \mu\Big(p^{-1}\Big(\{1\}\Big)\Big) = 0$. A subset $S \subset L^{p(\cdot)}(\Omega)$ is relatively weakly compact if and only if it is norm bounded and*

$$\lim_{n\to\infty} \sup_{f\in S} \frac{1}{(1+\lambda)^n} \int_\Omega |\lambda^n|^{p(t)} |f(t)|^{p(t)} d\mu = 0 \quad \textit{for} \quad \lambda \neq -1 \tag{2}$$

Proof. We prove by contradiction. Let $S \subset L^{p(\cdot)}(\Omega)$ be norm bounded and clearly suppose $S \subset B_{L^{p(\cdot)}}$. Hence, for every $f \in S$, we have $\int_\Omega |f(t)|^{np(t)} d\mu \leq 1$. Suppose that (2) does not hold, so there exist $\varepsilon > 0$, $(\lambda_n) \searrow 0$ and $(f_n) \subset S$ such that, for every $n \in \mathbb{N}$,

$$\int_\Omega \Big|\lambda_n f_n(t)\Big|^{np(t)} d\mu \geq (\lambda_n + 1)\varepsilon \tag{3}$$

Since $p_- = 1$ and $\mu(\Omega_1) = 0$, we can take a sequence $(n\delta_n) \searrow 0$ such that the sets $B_n = \{t \in \Omega : np(t) \leq n\delta_n\}$ satisfy $0 < \mu(B_n) \leq \frac{\varepsilon}{3n}$. Suppose that (λ_n) satisfies the following properties:

$$0 \leq \lambda_n + 1 \leq \frac{1}{2n}, \quad \sum_n \lambda_n \leq 1, \quad \sup_{t\in B_n^c} \frac{(n\lambda_n)^{np(t)}}{\lambda_n} \leq \frac{(n\lambda_n)^{n\delta_n}}{\lambda_n} \leq \frac{\varepsilon}{3} \tag{4}$$

where $B_n^c = \{t \in \Omega : np(t).\delta_n\}$. Consider the function $g_n(t) = \Big|\lambda_n f_n(t)\Big|^{n(p(t)-1)}$. For a.e. $t \in \Omega$, we have

$$2\Big|\lambda_n f_n(t) g_n(t)\Big| = 2\Big|\lambda_n f_n(t)\Big|^{np(t)}$$

Put $p(t) \cdot^* (t) - p^*(t)$, we get

$$\Big|g_n(t)\Big|^{np^*(t)} = \Big|\lambda_n f_n(t)\Big|^{np(t)}$$

Let $g(t) = \sup_n |g_n(t)|$. We claim that $g \in L^{p^*(\cdot)}(\Omega)$. Since $g^{np^*(\cdot)} \leq \sum_n g_n^{np^*(\cdot)}$, we get

$$\int_\Omega \Big|g(t)\Big|^{np^*(t)} d\mu \leq \sum_{n\in\mathbb{N}} \int_\Omega \Big|g_n(t)\Big|^{np^*(t)} d\mu = \sum_{n\in\mathbb{N}} \int_\Omega \Big|\lambda_n f_n(t)\Big|^{np(t)} d\mu$$
$$\leq \sum_{n\in\mathbb{N}} \lambda_n^{np^-} \int_\Omega \Big|f_n(t)\Big|^{np(t)} d\mu \leq \sum_{n\in\mathbb{N}} \lambda_n \leq 1$$

Considering the sets $A_n := \Big\{t \in \Omega : |f_n(t)| > n\Big\}$, we get

$$\inf_{t\in B_n} n^{np(t)} \mu(A_n) \leq \int_{B_n} \Big|f_n(t)\Big|^{np(t)} d\mu \leq 1$$

So, $\lim_{n\to\infty} \sup_{t\in A_n} \frac{1}{n^{np(t)}} \longrightarrow 0 \geq \mu(A_n)$. Therefore there exists n_0 such that $n > n_0$,

$$\int_{A_n} \Big|f_n(t)g(t)\Big| d\mu < \frac{\varepsilon}{6}$$

Hence

$$\int_\Omega \Big|\lambda_n f_n(t)\Big|^{np(t)} d\mu = \int_{A_n} \Big|\lambda_n f_n(t)\Big|^{np(t)} d\mu + \int_{A_n^c} \Big|\lambda_n f_n(t)\Big|^{np(t)} d\mu$$
$$\leq \int_{A_n} \Big|\lambda_n f_n(t)\Big|^{np(t)} d\mu + \sup_{t\in B_n^c} \Big(n(\lambda_n+1)\Big)^{np(t)} \mu\Big(A_n^c \cap B_n^c\Big)$$
$$+ \sup_{t\in B_n} \Big(n\lambda_n\Big)^{np(t)} \mu\Big(A_n^c \cap B_n\Big)$$
$$\leq \int_{A_n} 2\Big|\lambda_n f_n(t) g_n(t)\Big| d\mu + (\lambda_n+1)\frac{\varepsilon}{3} + n(\lambda_n+1)\frac{\varepsilon}{3n}$$
$$\leq 2\lambda_n \int_{A_n} \Big|f_n(t)g_n(t)\Big| d\mu + (\lambda_n+1)\frac{2\varepsilon}{3}$$
$$< (\lambda_{n+1})\varepsilon$$

which contradicts (2).
Conversely, let $g \in L^{p^*(\cdot)}(\Omega)$ and $r > 0$ such that $\int_\Omega |rg(t)|^{np^*(t)} d\mu < \infty$ or $\sup_n \int_\Omega |rg(t)|^{np^*(t)} d\mu \leq M < \infty$. Now given $\varepsilon > 0$, then there exists $\xi_0 > 0$ such that

$$\sup_{f\in S} \frac{1}{(1+\xi_0)^n} \int_\Omega |\xi_0 f(t)|^{np(t)} d\mu < \frac{\varepsilon r}{2}$$

Take $\delta > 0$ such that, for every measurable set E with $\mu(E) < \delta$,

$$\int_E |rg(t)|^{np^*(t)} d\mu \leq \frac{\varepsilon(\xi_0+1)r}{2}$$

Thus, using Young inequality ([5] Lemma 3.2.20), we have

$$\sup_{f\in S} \int_E |f(t)g(t)| d\mu \leq \frac{1}{(\xi_0+1)r}\Big[\sup_{f\in S} \int_E |\xi_0 f(t)|^{np(t)} d\mu + \int_E |rg(t)|^{np^*(t)} d\mu\Big]$$
$$\leq \frac{1}{r}\Big(\frac{\varepsilon r}{2}\Big) + \frac{1}{(\xi_0+1)r}\Big(\frac{\varepsilon(\xi_0+1)r}{2}\Big) = \varepsilon$$

We conclude that S is relatively weakly compact. $\square$

4 Some Results on Banach-Saks Property

Recall the following definitions describing the criteria for variable exponent spaces $L^{p(\cdot)}(\Omega)$, with $p^+ < \infty$ to be weakly Banach-Saks.

Definition 4.1. Let X be a Banach space over a field $\mathbb{R}$. Then X is said to be Banach-Saks if for every bounded sequence $(\xi_n) \subset X$ there exists a subsequence $(\xi_{n_k})_{k\in\mathbb{N}}$ which is Cesàro convergent. That is, there exists a point $\xi \in X$ such that

$$\left\| \frac{1}{k} \sum_{\alpha=1}^{k} \xi_{n_\alpha} - \xi \right\|_X < \varepsilon \quad \forall \quad \alpha > \alpha_0 \in \mathbb{N}$$

Equivalently,

$$\lim_{k\to\infty} \left\| \frac{1}{k} \sum_{\alpha=1}^{k} \xi_{n_\alpha} - \xi \right\|_X = 0.$$

Definition 4.2. A Banach space X is said to be weakly Banach-Saks if for every weakly convergent (ξ_n) there exists (ξ_{n_k}) which is Cesàro convergent. Every Banach-saks space is also weakly Banach-saks. The notion of Banach-Saks is hereditary. This means a Banach-Saks space (or weakly Banach-Saks space) passes the property to closed subspaces of themselves.

Theorem 4.3. *The space $L^{p(\cdot)}(\Omega)$ is weakly Banach-Saks if and only if $p^+ < \infty$.*

Proof. Let $\operatorname{ess}\inf_{x\in\Omega} p(x) < \infty$. Then $L^{p(\cdot)}(\Omega)$ has an isomorphic copy of l^∞ which is weakly Banach-Saks, so is $L^{p(\cdot)}(\Omega)$.
Assume on the other hand that the space $L^{p(\cdot)}(\Omega)$ is weakly Banach-Saks, Then we show that $p^+ < \infty$. To do this, let (ξ_n) be a pairwise disjoint weakly convergent sequence in $L^{p(\cdot)}(\Omega)$ which means we have a subsequence (ξ_{n_k}) which satisfies the condition of Cesàro convergence. Let $(\xi_n\chi_{\Omega_1})$ and $(\xi_n\chi_{\Omega\setminus\Omega_1})$ be weakly convergent sequences in $L^1(\Omega_1)$ and $L^{p(\cdot)}(\Omega\setminus\Omega_1)$ respectively. Since $L^1(\Omega_1)$ is weakly Banach-saks, then there exists a subsequence $(\xi_{n_k}\chi_{\Omega_1})$ which is Cesàro convergent. On other hand as

$$\begin{aligned}
\left\| \frac{1}{n} \sum_{\alpha=1}^{n} \xi_\alpha \right\|_{p(\cdot)} &= \left\| \frac{1}{n} \left[\sum_{\alpha=1}^{n} \xi_\alpha \chi_{\Omega_1} + \sum_{\alpha=1}^{n} \xi_\alpha \chi_{\Omega\setminus\Omega_1} \right] \right\|_{p(\cdot)} \\
&\le \left\| \frac{1}{n} \sum_{\alpha=1}^{n} \xi_\alpha \chi_{\Omega_1} \right\|_{p(\cdot)} + \left\| \frac{1}{n} \sum_{\alpha=1}^{n} \xi_\alpha \chi_{\Omega\setminus\Omega_1} \right\|_{p(\cdot)}
\end{aligned}$$

It remains to show that $(\xi_n\chi_{\Omega\setminus\Omega_1})$ is Cesàro convergent for some (ξ_{n_l}). Assume that (ξ_n) in $L^{p(\cdot)}(\Omega\setminus\Omega_1)$ the sequence (ξ_n) is relatively compact since it is weakly convergent.
Thus,

$$\lim_{\lambda\to 0} \inf_{n\in\mathbb{N}} \frac{1}{(1+\lambda)^n} \int_\Omega \left| \lambda\xi_k(t) \right|^{np(t)} dt = \lim_{\lambda\to 0} \inf_{n\in\mathbb{N}} \frac{\rho_{np(\cdot)(\lambda\xi_k)}}{(1+\lambda)^n} = 0 \quad \text{for} \quad \lambda \neq -1$$

Therefore,

$$\begin{aligned}
0 \ge -\lim_{n\to\infty} \rho_{np(\cdot)}\Big(\frac{1}{n}\sum_{\alpha=1}^{n} \xi_\alpha\Big) &= -\lim_{n\to\infty} \sum_{\alpha=1}^{n} \rho_{np(\cdot)}\Big(\frac{\xi_\alpha}{n}\Big) \le -\lim_{n\to\infty} \sum_{\alpha=1}^{n} \sup_{m\in\mathbb{N}} \rho_{np(\cdot)}\Big(\frac{\xi_m}{n}\Big) \\
&\le -\lim_{n\to\infty} \sum_{\alpha=1}^{n} \inf_{m\in\mathbb{N}} \rho_{np(\cdot)}\Big(\frac{\xi_m}{n}\Big) = -\lim_{n\to\infty} \inf_{m\in\mathbb{N}} \Big(1 \cdot \rho_{np(\cdot)}\Big(\frac{\xi_m}{n}\Big)\Big) \\
&= 0
\end{aligned}$$

This shows that $\operatorname{ess}\inf_{x\in\Omega} p(x) < \infty$. $\square$

Consider the following results involving the embeddings and closed subspaces of $L^{p(\cdot)}(\Omega)$.

Theorem 4.4. *Let $p(\cdot) \leq q(\cdot)$ and $p^+, q^+ < \infty$. Then $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is weakly Banach-saks if and only if $L^{p(\cdot)}(\Omega)$ is weakly Banach-saks.*

Theorem 4.5. *Let $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ be a closed variable exponent subspace such that $\mu(\Omega) \neq 0$ for $p(\cdot) \leq q(\cdot)$. Assume that $L^{p(\cdot)}(\Omega)$ is weakly Banach-Saks. Then the quotient space, $L^{p(\cdot)}(\Omega)\Big/L^{q(\cdot)}(\Omega)$ is weakly Banach-Saks.*

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