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A NEW CHARACTERIZATION OF CHEVALLEY GROUPS $G_2(2^n)$ BY $NSE(G_2(2^n))$

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Abstract

One of the important problems in finite group theory, is characterization of groups by specific property. For this purpose, in this paper, we prove that chevalley groups $G_2(2^n)$, where $2^{2n} + 2^n + 1$ is a prime number can be uniquely determined by $nse(G_2(2^n))$.

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1 Introduction

Let G be a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of orders of elements in G. If $k \in \pi_e(G)$, then we denote the set of the number of elements of order k in G by $m_k(G)$ and the set of the number of elements with the same order in G by $nse(G)$. In otherwords, $nse(G) = \{m_k(G) \ : \ k \in \pi_e(G)\}$. Also we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $n_p(G)$. If a ,b be two integer numbers, then we denote the greatest common divisor of a, b by $(a; b)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$.
The characterization of groups by $nse(G)$ pertains to Thompson's problem ([21]). Thompson's Problem. Let

$$\Gamma(G) = \{(n, m_n) \mid n \in \pi_e(G) \ and \ m_n \in nse(G)\},$$

where m_n is the number of elements with order n. Suppose that $\Gamma(G) = \Gamma(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

The authors in ([7],[8],[9],[10],[11],[12],[13],[14]) proved that these groups can be characterized by using the set of elements with the same order and order of the group. Furthermore in the way, characterization of group by $nse(G)$ in ([1],[2],[3],[20],[22]) the authors proved that some of groups are characterizable by the number of elements with the same order. For example, some linear groups, symmetric groups, $PSL(3,q)$, $G_2(q)$, where q^2+q+1 such that q is odd, $q \equiv -1 (\mod 3)$ and $L_2(3^n)$ by using $nse(G)$ can be characterized. In this paper, we prove that chevalley groups $G_2(2^n)$, where $2^{2n}+2^n+1$ is a prime number can be uniquely determined by number of elements with the same order. In fact, we prove the following main theorem.

Main Theorem. Let G be a group with $nse(G) = nse(G_2(2^n))$ where $2^{2n}+2^n+1$ is a prime number. Then $G \cong G_2(2^n)$.

Lemma 1.1. *[16] Let G be a Frobenius group of even order with kernel K and complement H. Then*

1. *$t(G) = 2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*
2. *$|H|$ divides $|K| - 1$;*
3. *K is nilpotent.*

Definition 1.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 1.3. *[5] Let G be a 2-Frobenius group of even order. Then*

1. *$t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ and $\pi(K/H) = \pi_2$;*
2. *G/K and K/H are cyclic groups satisfying $|G/K|$ divides $|Aut(K/H)|$.*

Lemma 1.4. *[29] Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

1. *G is a Frobenius group;*
2. *G is a 2-Frobenius group;*
3. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|Out(K/H)|$.*

Lemma 1.5. *[15] Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 1.6. *Let G be a group containing more than two elements. If the integer number s be the maximal numbers of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2-1)$.*

Proof. You see([24]). □

Lemma 1.7. *Let G be a finite group. Then for every $i \in \pi_e(G)$, $\varphi(i)$ divides $m_i(G)$, and i divides $\sum_{j|i} m_j(G)$. Moreover, if $i > 2$, then $m_i(G)$ is even.*

Proof. By Lemma 1.5, the proof is straightforward. □

Lemma 1.8. *[30] Let q, k, l be natural numbers. Then*

1. $(q^k - 1, q^l - 1) = q^{(k,l)} - 1$.
2. $(q^k + 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$
3. $(q^k - 1, q^l + 1) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$ the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Lemma 1.9. *[26] Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:*

(a) $L_2(q)$, $q \equiv \pm 2 \pmod 5$;

(b) $L_3(q)$, $q \equiv \pm 2 \pmod 5$;

(c) $G_2(q)$, $q \equiv \pm 2 \pmod 5$;

(d) $U_3(q)$, $q \equiv \pm 2 \pmod 5$;

(e) ${}^3D_4(q)$, $q \equiv \pm 2 \pmod 5$;

(f) ${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$.

Lemma 1.10. *[18] Let p, q be prime numbers and m,n be natural numbers such that $p^m - q^n = 1$. Then one of the following statements holds:*

1. *If $m = 1$ then $p = 2^{2^t} + 1$ where $t \geq 0$ is a integer number;*
2. *If $n = 1$ then $q = 2^{p_0} - 1$,where p_0 is a prime number;*
3. *If $m, n > 1$then $(p, q, m, n) = (3, 2, 2, 3)$;*

2 Proof of the Main Theorem

In this section, we prove that the chevalley groups $G_2(2^n)$ are characterizable by the number of elements with the same order. In fact, we prove that if G is a group with $nse(G) = nse(G_2(2^n))$, where $2^{2n} + 2^n + 1$ is a prime number, then $G \cong G_2(2^n)$. We divide the proof to several lemmas. From now on, we denote the group $G_2(2^n)$ by R and the numbers 2^n and $2^{2n} + 2^n + 1$ by q and p, respectively. Recall that G is a group with $nse(G) = nse(R)$. Let G be a group such that $nse(G) = nse(G_2(2^n))$, where $2^{2n} + 2^n + 1$ is a prime number, and m_n be the number of elements of order n. By Lemma 1.6 we have that G is finite. We note that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then mn is even. If $n \in \pi_e(G)$, then by Lemma 1.7 and the above discussion, we have

$\phi(n) \mid m_n$,

$n \mid \sum_{d|n} m_d$

Lemma 2.1. *p is an isolated vertex of $\Gamma(G)$.*

Proof. We prove that p is an isolated vertex of $\Gamma(G)$. Assume the contrary, then there is a prime number $t \in \pi(G) - \{p\}$, so that $tp \in \pi_e(G)$. So, we deduce $tp \geq 2p = 2(q^2+q+1) \geq q^2+q+1$, so we deduce $k(G) > q^2+q+1$, which is a contradiction. Hence p is an isolated vertex, $t(G) \geq 2$. □

Lemma 2.2. *If $rp \notin \pi_e(G)$, for every $r \neq p \in \pi(G)$, then $p \mid m_r$.*

Proof. By ([19, Theorem1]) the maximal torus T of $G_2(2^n)$ have the orders q^2+q+1 and q^2-q+1. Then, there is an element $x \in R$ and some torus T such that $|x| = r$ and $T \leq C_G(x)$ for some T. It follows that so $|cl(x)|$ is the multiple of $\frac{|R|}{|T|}$ for some T. But $m_r(R) = \sum_{|x|=r, x\neq 1} |cl(x)|$. Hence $p \mid m_r$. □

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Lemma 2.3. $m_p(G) = m_p(R) = \frac{q^7(q^2-1)^2(q^3+1)}{6}$ *and* $n_p(G) = \frac{|R|}{6p}$.

Proof. First we know that $|R| = q^6(q^6-1)(q^2-1)$. Since $|R_p| = p$, we deduce that R_p is a cyclic group of order p. Thus $m_p(R) = \phi(p)n_p(R) = p-1)n_p(R)$. Now it is enough to show $n_p(R) = \frac{|R|}{6p}$. By[29], p is an isolated vertex of $\Gamma(G)$. Hence $|C_R(R_p)| = p$ and $|N_R(R_p)| = xp$ for a natural number x. We know that $\frac{N_R(R_p)}{C_R(R_p)}$ embed in $Aut(R_p)$, which implies $x \mid p-1$. Furthermore, by Sylow's Theorem, $n_p(R) = |R : N_R(R_p)|$ and $n_p(R) \equiv 1($ mod $p)$. Therefore p divides$\frac{|R|}{(xp)} - 1$. Thus q^2+q+1 divides $\frac{q^6(q^6-1)(q^2-1)}{xp} - 1$. It follows that q^2+q+1 divides $\frac{q^{14}-q^{12}-q^8+q^6}{q^2+q+1} - x$. As a result $q^2+q+1 \mid q^{12}-q^{11}-q^{10}+2q^9-q^8-q^7+q^6-x$, so we have $q^2+q+1 \mid (q^2+q+1)(q^{10}-2q^9+4q^7-5q^6+6q^4-6q^3+6q-6)+(6-x)$ we have $p \mid 6-x$. Since $x \mid p-1$, as a result $x \mid 2^{2n}-2n$, we deduce that $x = 6$. It followes $n_p(R) = \frac{|R|}{6p}$. □

Lemma 2.4. $|G|$ *divides* $\frac{(q^2+q)|R|}{6}$.

Proof. By Lemma 2.2, we have $rp \notin \pi_e(G)$ for any prime $r \in \pi(G) - \{p\}$. It follows that the sylow r-subgroup G_r of G acts fixed freely on the set of elements of order p and so $|G_r| \mid m_p$. Therefore $|G| \mid \frac{(q^2+q)|R|}{6}$. □

Lemma 2.5. $m_2(G) = m_2(R)$. *In particular* $p \mid m_2(R)$.

Proof. First if $2 < n \in \pi_e(G)$, then m_n is even. By Lemma 1.7 $2 \mid 1 + m_2(R)$. On the other hand, by ([4],[20]) in G the only odd number in $nse(G) - 1$ is $m_2(G)$. Hence we have $m_2(G) = m_2(R)$. By Lemma 2.2 we have $p \mid m_2(R)$. □

Lemma 2.6. *The group G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.6, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Lemma 2.1, p is an isolated vertex of $\Gamma(G)$. Thus we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K|-1$, we conclude that the last case can not occur. So $|H| = p$ and $|K| = |G|/p$, hence $q^2+q+1 \mid \frac{q^6(q^6-1)(q^2-1)}{q^2+q+1} - 1$. So we conclude $q^2+q+1 \mid (q^{10} - 2q^9 + 4q^7 - 5q^6 + 6q^4 - 6q^3 + 6q - 6) + 5$. Thus $p \mid 5$ which is impossible. □

Lemma 2.7. *The group G is not a 2-Frobenius group.*

Proof. We prove that G is not a 2-Frobenius group. On the opposite, assume G be a 2-Frobenius group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups by kernels K/H and H respectively. Set $|G/K| = x$. Since p is an isolated vertex of $\Gamma(G)$, then $\pi_2(G) = \{p\}$ it follows that $|K/H| = p$. Now since $|G/K|$ divides $|Aut(K/H)|$, we deduce that $|G/K| \mid p-1$. By Lemma 1.8 we have $(p-1, q-1) = 1$. Thus $t \mid |H|$, where $t = q-1$ now since that H is nilpotent.So $H_t \rtimes K/H$ is a Frobenius group with kernel H_t and complement K/H. So $|K/H|$ divides $|H_t| - 1$. It implies that $q^2 + q + 1 \leq (q-1) - 1$, but this is a contradiction. □

Lemma 2.8. *The group G is isomorphic to the group R.*

Proof. By Lemma 2.1, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 1.4. Now Lemma 2.6 and Lemma 1.3 implies that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of Lemma 1.4 occure. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. On the other hand, we know that $5 \nmid |G|$. Thus K/H is isomorphic to one of the groups in Lemma 1.9. Hence we consider the flowing cases:

Step 1. If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2m+1}$, then by [29], $\pi({}^2G_2(q')) = q' \pm \sqrt{3q'} + 1$. For this purpose, we consider $q^2+q+1 = q' \pm \sqrt{3q'}+1$. It follows that $2^{2n}+2^n+1 = 3^m(3^m \pm 1)$ as a result $2^n(2^n+1) = 3^m(3^m \pm 1)$. Since $(2^n, 2^n+1) = 1$, so we deduce $2^n(2^n+1) = 3^m(3^m+1)$ and also $2^n(2^n+1) = 3^m(3^m-1)$. For this purpose if $2^n = 3^m+1$, then by Lemma 1.10 we deduce $m = 2$, $n = 3$. Since $|{}^2G_2(243)| \nmid |G_2(8)|$ we deduce a contradiction.The other case is a contradiction, similarily.

Step 2. Suppose that $K/H \cong {}^3D_4(q')$, where $q' \equiv \pm 2 \pmod 5$. Then by[29], $\pi({}^3D_4(q')) = q'^4 - q'^2 + 1$. So we consider $q^2+q+1 = q'^4 - q'^2 + 1$, in result $q(q+1) = q'^2(q'^2-1)$. Now since that $(q, q+1) = 1$, so we deduce $q = q'^2 - 1$. Now since $|{}^3D_4(q')| \nmid |G|$, which is a contradiction.

Step 3. If $K/H \cong U_3(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [19, 29], $\pi(U_3(q') = (q'^2 - q' + 1)/(3, q'+1)$. If $(3, q'+1) = 1$, then we consider $q^2+q+1 = q'^2 - q' + 1$ in result $q(q+1) = q'(q'-1)$. Now since $(q, q+1) = (q', q'-1) = 1$, we deduce $q' = q+1$. But $|U_3(q')| \nmid |G|$, where this is a contradiction. Now we assume $(3, q'+1) = 3$ then we consider $(q^2+q+1 = q'^2 - q' + 1)/3$, so $3q(q+1) = (q'-2)(q'+1)$. Now we deduce that

$(q'-2, q'+1)=3$. Since $(3, q'+1)=3$. So $q'+1=q+1$, $q'-2=3q$ or $q'+1=3q$, $q'-2=q+1$. First we suppose $q'+1=3q$, $q'-2=q+1$ then we deduce $q'=q$, $q'=3q+2$. It followes $q=1$, in other words $2^n=1$, which this is a contradiction. Now if $q'+1=3q$, $q'-2=q+1$, then we have $q'=3q-1$, $q'=q+3$. As a result $q=2$ so $2^n=2$, $q'=5$ then $n=1$. Now $|U_3(5)| \nmid |G_2(2)|$, which is impossible.

Step 4. If $K/H \cong L_3(q')$, where $q' \equiv \pm 2 \pmod 5$, then for this purpose we consider two cases. First we assume $(3, q'-1)=1$ then we have $q^2+q+1=q'^2+q'+1$. As a result $q(q+1)=q'(q'+1)$ now since $(q, q+1)=(q', q'+1)=1$ we deduce $q'=q$. So $q'=2^n$. On the other hand, we know $2^n \equiv 2(\mod 3)$ hence $q' \equiv 2(\mod 3)$, but this is contrary, because $q' \equiv \pm 2(\mod 5)$. So we have a contradiction. Now if $(3, q'-1)=3$ then $q^2-q+1=\frac{q'^2+q'+1}{3}$. It follows that $3q^2+3q+3=q'^2+q'+1$. As a result $3q^2+3q=q'^2+q'-2$ so $3q(q+1)=(q'-1)(q'+2)$. Since $(q'-1, q'+2)=3$, so $q'-1=q+1$, $q'+2=3q$ or $q'-1=3q$, $q'+2=q+1$. First, we suppose $q'-1=q+1$, $q'+2=3q$ then we deduce $q'=q+2$, $q'=3q-2$. As a result we have $q=2$, $q'=4$. In other words $n=1$, now since $|L_3(4)| \nmid |G_2(2)|$, so we have a contradiction. Now we consider the other case, if $q'-1=3q$, $q'+2=q+1$, then we have $q'=3q+1$, $q'=q-1$. It followes that $q=-1$ or $2^n=-1$, which is impossible.

Step 5. If $K/H \cong L_2(q')$ where $q' \equiv \pm 2 \pmod 5$, $q'=p'^m$ then first we assume q' be even, then $p=q' \pm 1$. So we have $q^2+q+1=q' \pm 1$. First we assume $q^2+q+1=q'+1$ then $q(q-1)=q'$, which is a contradiction because q' is power of p'. Now if $q^2+q+1=q'-1$ then $q^2+q+2=q'$. Since$|L_2(q')| \nmid |G|$, is a contradiction. In the way we assume q' be odd. First we consider $p=q'$, then we have $q^2+q+1=q'$, now since $|L_2(q')| \nmid |G|$, so we have a contradiction. Now if $p=\frac{q' \pm 1}{2}$, then we have $q^2+q+1=\frac{q' \pm 1}{2}$, so $q'=2q^2+2q+1$ or $q'=2q^2+2q+3$. Since $|L_2(q')| \nmid |G|$, we have a contradiction.

Hence, we deduce $K/H \cong R$, then $|K/H|=|R|$. Since p is an isolated vertex and also $p \mid |K/H|$, so we consider $q^2+q+1=q'^2+q'+1$, then we deduce $q=q'$, as a result $n=n'$. On the other hand, since $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we have $H=1$, $G=K \cong R$ and the proof is complete. □

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