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Multiplication injective S-act on monoid

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Abstract

In this paper, we define a new kind of injectivity, namely multiplication injective S-act with respect to inclusion into multiplication S-act on monoid S. We study the product and coproduct of multiplication injective S-acts. Also, we investigate the Skornjakhoph' Theorem for multiplication injective S-act.

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1 Introduction and preliminaries

There are many research on kind of injectivity such as C-injectivity, CC-injectivity, InC-injectivity. For more see, [4], [6] [8], [2], [3], [5] and [9].

We define multiplication injective S-act and investigate some properties of this notation and study the behavior of multiplication injective S-act with respect to the product, coproduct. Also, we investigate Skornjakhoph' s Theorem for multiplication injective S-acts.

First, we give some preliminaries needed in the sequel. Let S be a monoid. By a (*right*) *S-act* or *act over* S, we mean a non-empty set A together with a map $A \times S \to A$, $(a, s) \mapsto as$, such that for all $a \in A, s, t \in S$, $(as)t = a(st)$ and $a1 = a$. A non-empty subset $B \subseteq A$ is called a *subact* of A if $bs \in B$ for all $b \in B$ and $s \in S$. An element $\theta \in A$ for which $\theta s = \theta$ for all $s \in S$ is said to be a *zero* or *fixed element* of A. Clearly, S is an S-act with the operation as the action. Let A and B be two S-acts. A mapping $f : A \to B$ is called an S*-homomorphism* if $f(as) = f(a)s$ for all $a \in A, s \in S$. The category of all S-acts as well as all homomorphisms between them is denoted by **Act-**S. A non-empty subset I of a monoid S is called a *right ideal* of S if $xs \in I$ for any $x \in I$ and $s \in S$, and it is called two-sided ideal if $xs, sx \in I$ for any $x \in I$ and $s \in S$. An S-act A is called an injective S-act, if for any monomorphism $f : B \longrightarrow C$ and S-homomorphism $g : B \longrightarrow A$, there exists an S-homomorphism $h : C \longrightarrow A$ such that $hf = g$. If injective S-act is with respect to the inclusion $B \subseteq C$, we call A is injective relative to inclusion $A \subseteq B$. For more about S-acts see [1]. We recall that an S-act A is called a multiplication S-act if for any subact B of A, there exist a two-sided ideal I of S such that $B = AI$. Every S-homomorphism image of multiplication S-act is a multiplication S-act and, any pure subact of multiplication S-act is multiplication. Also, on commutative monoid, any cyclic S-act is a multiplication S-act.

2 multiplication injective S-act

In this section, we present the definition of injective multiplication S-act and study some properties of this notation.

Definition 2.1. An S-act A is called multiplication injective S-act if for any inclusion $\iota : B \hookrightarrow C$ from subact B to multiplication S-act C and S-homomorphism $f : B \longrightarrow C$, there exists an S-homomorphism $g : C \longrightarrow A$ such that $g\iota = f$. Clearly, any injective S-act is multiplication injective S-act, and the conversely is not always true. For example, any S-act without zero on group S is multiplication injective, and it is not injective. Now, consider monoid $S = \{0, 1\}$. Clearly, any S-act is a multiplication injective S-act.

We recall that from[6], an S-act A is said to be *indecomposable codomain injective* or *InC-injective* for short, if it is injective with respect to all embeddings into indecomposable acts. By [6, Corollary 2.8], an S-act is InC-injective if and only if it is injective relative to all embeddings into cyclic acts. Then, using Theorem 2.3, we get the following:

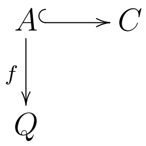
Corollary 2.2. *Let S be a commutative monoid. Any InC-injective S-act is a multiplication injective S-act.*

We recall that an S-act is weakly injective if it is injective relative to all embedding of ideals to monoid S. Obviously, on commutative monoid, any multiplication injective S-act is weakly injective.

The Skornjakhoph' Theorem is provided that any S-act with zero is injective if and only if it is injective relative from subact to cyclic S-act. Now, in the following, we study the Skornjakhoph' Theorem for multiplication injective S-acts.

Theorem 2.3. *An S-act Q is a multiplication injective S-act if and only if it is injective relative to all inclusions into multiplication cyclic S-act.*

Proof. The necessity is clear. For sufficiency, consider the following diagram,



We claim there exists an S-homomorphism $g : C \longrightarrow Q$ such that $g|_A = f$. Let $\Sigma = \{(X,h)|\ X$ is a subact of C, $A \subseteq X \subseteq C, h : X \longrightarrow Q$ is an S-homomorphism extending $f\}$. Clearly, $(A,f) \in \Sigma$, so Σ is not the empty set. Consider the following partial order on Σ,

$$(X_1,h_1) \leq (X_2,h_2) \Leftrightarrow X_1 \subseteq X_2 \text{ and } h_2|_{X_1} = h_1.$$

For any chain $\{(X_i,g_i)_i\}_{i\in I}$ in Σ, the pair $(\bigcup_{i\in I} X_i, \overline{h})$ where $\overline{h}(x_i) = h_i(x_i)$ for $x_i \in X_i$ is an upper bound. By Zorn's Lemma, there exists a maximal element $(D,\hat{h})$ in Σ. We show that $D = C$, and so we have $\overline{f} = \hat{h}$ extends f. Suppose that $D \neq C$. So there exists $c \in C \setminus D$. Since C is a multiplication S-act, there exists a two-sided ideal I of S such that $D = CI$. Obviously, $D \cap cS \neq \emptyset$. Let $H := D \cap cS$ and $h := \hat{h}|_H$

It follows from the hypothesis that there exists an S-homomorphism $k : cS \to A$ such that $k|_H = h$. Set $E := D \cup cS$. Define $l : E \to A$ by

$$l(x) = \begin{cases} \hat{h}(x) & x \in D \\ k(x) & x \in cS \end{cases}$$

for every $x \in E$. Since $\hat{h}|_H = h = k|_H$, l is well-defined and clearly an S-homomorphism. Also $l|_B = \hat{g}|_B = f$. Now, we have $(E,l) \in \Sigma$, which contradicts by the maximality of $(D,\hat{h})$.

□

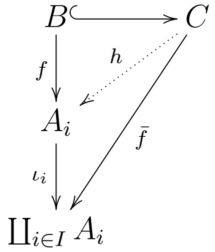
Corollary 2.4. *Let S be a comutative monoid. Any S-act with zero is injective if and only if it is a multiplication injective.*

Proposition 2.5. *Let $\{A_i \mid i \in I\}$ be a family of S-acts. Then the product $\prod_{i\in I} A_i$ is a multiplication injective S-act if each A_i is a multiplication injective S-act. The converse also holds if each A_i has a fixed element.*

Proof. See [7, Theorem 3.24]. □

Proposition 2.6. *Let $\{A_i \mid i \in I\}$ be a family of S-acts. If the coproduct $\coprod_{i\in I} A_i$ is multiplication injective S-act, then so is each A_i.*

Proof. Assume that $\coprod_{i\in I} A_i$ is a multiplication injective S-act. Let $i \in I$. We show that A_i is a multiplication injective S-act. Let B be a subact of C and consider the diagram



where f is an S-homomorphism and ι_i is the canonical injection. Since $\coprod_{i\in I} A_i$ is a multiplication injective S-act, there exists an S-homomorphism $\bar{f} : C \longrightarrow \coprod_{i\in I} A_i$ such that $\bar{f}|_B = \iota_i f = f$. We claim that $\mathrm{Im}\bar{f} \subseteq A_i$. Let there exist $x \in C$ and $j \in I, j \neq i$, such that $\bar{f}(x) \in A_j$. Since C is a multiplication S-act, there exists a two-sided ideal I of S such that $B = CI$. So $\bar{f}(xi) = \iota_i f(xis) = f(xi) \in A_i$, $i \in I$. On the other hand, $\bar{f}(xi) = \bar{f}(x)i \in A_j$. Then $\bar{f}(xi) \in A_i \cap A_j$ which is a contradiction. Now considering $h := \bar{f} : C \to A_i$, we get $h|_B = f$. □

Theorem 2.7. *The following statements hold for any monoid S:*

(i) *All coproducts of multiplication injective S-acts are multiplication injective.*

(ii) $\Theta \sqcup \Theta$ *is a multiplication injective S-act.*

(iii) *S is left reversible.*

Proof. The implication (i) ⇒ (ii) is trivial.

(ii) ⇒ (iii) By the same method to the proof of Proposition 2.12 (iii) ⇒ (iv) in [1] for the multiplication injective S-acts, the result is obtained.

(iii) ⇒ (i) Let A_i be a multiplication injective S-act for each $i \in I$. We apply Theorem 2.3 to prove that $\coprod_{i\in I} A_i$ is a multiplication injective S-act. Suppose that A is a subact of a multiplication cyclic S-act $B = bS$ and $f : A \to \coprod_{i\in I} A_i$ is an S- homomorphism. Also, consider the epimorphism $\pi := \lambda_b : S \to B$, the right ideal $K := \pi^{-1}(A)$ of S and $\tau := \pi|_K : K \to A$.

We claim that there exists $i \in I$ for which $\mathrm{Im}f \subseteq A_i$. Otherwise, $\mathrm{Im}f \cap A_i$ and $\mathrm{Im}f \cap A_j$ are non-empty for some $i, j \in I, i \neq j$, which clearly gives that $\mathrm{Im}f$ is a decomposable subact of $\coprod_{i\in I} A_i$. Using [1, Lemma 1.5.36], this implies that A and hence K is decomposable which contradicts the weak left reversibility of S. This gives the existence of $i \in I$ such that $\mathrm{Im}f \subseteq A_i$. Since A_i is a multiplication injective S-act, f can be extended to an S-homomorphism $\bar{f} : B \to A_i$. So, let $h := \iota_i \bar{f} : B \to \coprod_{i\in I} A_i$. So, we have $h|_A = f$. □

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