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A Type of Almost Co-Kähler Manifold Satisfying Vacuum Static Equation

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Abstract

In this paper we consider $(\tilde{k}, \tilde{\mu})$ almost co-Kähler manifold satisfying the vacuum static equation. First we prove that if a $(\tilde{k}, \tilde{\mu})$ almost co-Kähler manifold satisfies the vacuum static equation then its scalar curvature satisfies certain relation or the solution of the equation is trivial. Next we prove that the value of the scalar curvature is constant considering the fact that the vacuum static equation has non-trivial solution.

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1 Introduction

The study of vacuum static spaces ([25]) stands at the forefront of theoretical physics, offering a unique and intriguing perspective on the fundamental nature of the universe.

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One of the central motivations for studying vacuum static spaces stems from their role in advancing comprehension of general relativity, the cornerstone of modern gravitational theory. In 1915, Albert Einstein introduced the theory of General Relativity. According to this theory gravitational field is the spacetime curvature and its source is the energy-momentum tensor. A spacetime([4], [10], [14], [23]) of General Relativity is regarded as a 4-dimensional time-oriented Lorentzian manifold. The energy-momentum tensor describes the matter content of the spacetime. The matter content is assumed to be fluid having density, pressure and having dynamical and kinematic quantities like velocity, acceleration, vorticity, shear and expansion([1], [24]). The fluid is called perfect because of the absence of heat conduction terms and stress tensor corresponding to viscosity.
The field equation governing the perfect fluid motion is Einstein's gravitational equation

$$\kappa T(X,Y) = Ric(X,Y) + (\lambda - \frac{r}{2})g(X,Y), \tag{1}$$

where X, Y are smooth vector fields on M, κ is the gravitational constant and λ is the cosmological constant. Here T is the energy-momentum tensor given by

$$T(X,Y) = \rho g(X,Y) + (\sigma + \rho)\theta(X)\theta(Y),$$

where ρ is the isotropic pressure, σ is the energy density and θ is the 1-form given by $\theta(X) = g(X,\psi)$, where ψ is the velocity vector field of the fluid and $g(\psi,\psi) = -1$. Static spacetimes are important global solutions to the Einstein's equations.
Let (M^n, g) be an n-dimensional smooth Riemannian manifold. It is said to be a static space with perfect fluid if there exists a non-trivial smooth function $f : M \to \mathbb{R}$ such that

$$Ddf - f(Ric - \frac{r}{n-1}g) = \frac{1}{n}(\frac{r}{n-1} + \triangle f)g, \tag{2}$$

where Ddf is the Hessian of f, $\triangle$ is the negative Laplacian of f, Ric is the Ricci tensor and r is the scalar curvature. If the static space additionally satisfies the property $\frac{r}{n-1} + \triangle f = 0$ then it is termed as vacuum static space. For a vacuum static space (1.2) takes the form

$$Ddf - f(Ric - \frac{r}{n-1}g) = 0, \tag{3}$$

and this equation is called vacuum static equation. the exploration of vacuum static spaces has been a significant area of interest in theoretical physics and Mathematics particularly within the context of general relativity. Many Researchers have engaged in extensive studies to comprehend the theoretical implications and physical consequences of vacuum static solutions to Einstein's field equations. Recently Hawan and Yun([15]) consider vacuum static spaces with the complete divergence of the Bach tensor and Weyl tensor. Other interesting works in this regard can be found in [8], [12], [17], [22]. Motivated by the works of D. S. Patra et al.([3]) here we study vacuum static space on $(\tilde{k}, \tilde{\mu})$ almost co-Kähler manifold.
The paper is organized as follows. after the introduction in section 1, in section 2, we give some preliminaries and formulas of $(\tilde{k}, \tilde{\mu})$ almost co-Kähler manifold and of vacuum static spaces. In section 3, we prove certain results on $(\tilde{k}, \tilde{\mu})$ almost co-Kähler manifold satisfying vacuum static equation.

2 Preliminaries

This section consists of some basic definitions and properties of almost contact metric manifolds. A $(2n+1)$- dimensional smooth Riemannian manifold (M^{2n+1}, g) is said to be almost contact metric manifold if there exist on M a $(1,1)$ tensor field φ, a 1-form η and a vector field ξ such that

$$\varphi^2 X = -X + \eta(X)\xi, \;\; \varphi(\xi) = 0, \;\; \eta \circ \varphi = 0, \;\; \eta(\xi) = 1, \tag{4}$$

for any smooth vector field X on M. In an almost contact metric manifold we also have

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{5}$$

for all smooth vector fields X, Y on M.The fundamental 2-form Φ is given by $\Phi(X, Y) = g(X, \varphi Y)$. An almost contact metric manifold is said to be almost co-Kähler manifold ([2],[5], [6], [13], [16]) if $d\theta = 0$ and $d\Phi = 0$. An almost co-Kähler manifold is said to be normal if and only if

$$N^{(1)}(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0, \tag{6}$$

for any smooth vector fields X, Y in M, where N_φ denotes the Nijenhuis torson of φ, given by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{7}$$

A normal almost co-Kähler manifold is a co-Kähler manifold. On an almost co-Kähler manifold we consider three self adjoint operators

$$h = \frac{1}{2}\pounds_\xi \varphi, \; l = R(., \xi)\xi, \; h' = h \circ \varphi$$

where R is the Riemann curvature tensor of g. These three operators satisfy the following identities([18], [19], [21]):

$$trh = trh' = 0, \;\; h\xi = 0, \;\; h' = -\varphi h, \tag{8}$$

$$\nabla \xi = h', \; \nabla_\xi \varphi = 0, \; \nabla_\xi h = -h^2\varphi - \varphi l, \tag{9}$$

$$Ric(\xi, \xi) + trh^2 = 0, \tag{10}$$

$$\varphi l \varphi - l = 2h^2. \tag{11}$$

where ∇ is the Levi-Civita connection of g. The Ricci operator Q is given by $Ric(X, Y) = g(QX, Y)$.

If the curvature tensor R of an almost co-Kähler manifold satisfies

$$R(X, Y)\xi = \tilde{k}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)hX - \eta(X)hY), \tag{12}$$

foe all smooth vector fields X, Y on M and $\tilde{k}, \tilde{\mu} \in \mathbb{R}$ then it is called $(\tilde{k}, \tilde{\mu})$-Almost co-Kähler manifold. Using (11) and (12) we have

$$h^2 = \tilde{k}\varphi^2. \tag{13}$$

In $(\tilde{k}, \tilde{\mu})$-Almost co-Kähler manifold we have the following identities([20]):

$$\nabla_\xi h = \tilde{\mu} h' \tag{14}$$

$$\nabla_\xi h^2 = 0, \tag{15}$$

$$l\varphi - \varphi l = 2\tilde{\mu} h'. \tag{16}$$

Furthermore, in this manifold the tensor field h satisfies the following relation([11]):

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X =& \tilde{k}(\eta(Y)\varphi X - \eta(X)\varphi Y + 2g(\varphi X, Y)\xi) \\ &+ \tilde{\mu}(\eta(Y)\varphi h X - \eta(X)\varphi h Y),\end{aligned} \tag{17}$$

where X and Y are smooth vector fields on M. The Ricci operator Q of an almost co-Kähler manifold with $\tilde{k} < 0$ is given by ([9])

$$Q(Z) = \tilde{\mu} h(Z) + 2n\tilde{k}\eta(Z)\xi \tag{18}$$

for all smooth vector field Z on M.

Lemma 2.1. *([7], [12]) If a Riemannian metric g satisfies the vacuum static equation, then its scalar curvature is constant.*

In a $(2n+1)$-dimensional almost co-Kähler manifold the vacuum static equation (3) can be written as

$$\nabla_X Df = f\{QX - \frac{r}{2n}X\}, \tag{19}$$

where D is the gradient operator and X is any smooth vector field on M.
Differentiating (19) covariantly along an arbitrary vector field Y we obtain

$$\nabla_Y \nabla_X Df = (Yf)\{QX - \frac{r}{2n}X\} + f\{(\nabla_Y Q)X + Q(\nabla_Y X) - \frac{r}{2n}\nabla_Y X\}. \tag{20}$$

Using the equation (20) in the curvature formula $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ yields

$$\begin{aligned}R(X, Y)Df =& (Xf)QY - (Yf)QX + f\{(\nabla_X Q)Y - (\nabla_Y Q)X\} \\ &- \frac{r}{2n}\{(Xf)Y - (Yf)X\},\end{aligned} \tag{21}$$

for all smooth vector fields X, Y and Z on M.

3 Vacuum static equation in a $(\tilde{k}, \tilde{\mu})$ almost co-kähler manifold

In this section we consider $(\tilde{k}, \tilde{\mu})$ almost co-Kähler manifold satisfying vacuum static equation. Here we consider the case $h \neq 0$ and prove the following:

Theorem 3.1. *If a proper $(\tilde{k}, \tilde{\mu})$ almost co-Kähler manifold with $\tilde{k} < 0, r \neq 2n(2n+1)\tilde{k}$ satisfies the vacuum static equation then either the scalar curvature r satisfies $\{(2n+1)\tilde{k} - \frac{r}{2n})^2 + \tilde{\mu}^2\tilde{k}\} = 0$, or the equation cannot have any non-trivial solution.*

Proof. In a $(\tilde{k}, \tilde{\mu})$ almost co-Kähler manifold we have

$$R(X,Y)\xi = \tilde{k}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)hX - \eta(X)hY) \tag{22}$$

Taking inner product of both side of (22) with Df

$$g(R(X,Y)\xi, Df) = \tilde{k}(\eta(Y)Xf - \eta(X)Yf) + \tilde{\mu}(\eta(Y)(hX)f - \eta(X)(hY)f) \tag{23}$$

Substituting $X = \xi$ in (23) we obtain

$$g(R(\xi,Y)\xi, Df) = \tilde{k}(\eta(Y)\xi f - Yf) - \tilde{\mu}((hY)f) \tag{24}$$

Now consider (21)

$$\begin{aligned} R(X,Y)Df =&(Xf)QY - (Yf)QX + f\{(\nabla_X Q)Y - (\nabla_Y Q)X\} \\ &- \frac{r}{2n}\{(Xf)Y - (Yf)X\} \end{aligned} \tag{25}$$

Taking inner product with ξ in both side of (25) we obtain

$$\begin{aligned} g(R(X,Y)Df, \xi) =&2n\tilde{k}((Xf)\eta(Y) - (Yf)\eta(X)) - \frac{r}{2n}((Xf)\eta(Y) - (Yf)\eta(X)) \\ &+ f\{g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X)\} \end{aligned} \tag{26}$$

Using (19) and (17) in (26) we get

$$g(R(X,Y)Df, \xi) = (2n\tilde{k} - \frac{r}{2n})((Xf)\eta(Y) - (Yf)\eta(X)) - 2fg(X, \varphi Y) \tag{27}$$

Putting $X = \xi$ in the previous equation we obtain

$$g(R(\xi,Y)\xi, Df) = (\frac{r}{2n} - 2n\tilde{k})((\xi f)\eta(Y) - (Yf)) \tag{28}$$

Using of (23) and (28) gives us

$$(\frac{r}{2n} - 2n\tilde{k})\{(\xi f)\eta(Y) - (Yf)\} = \tilde{k}\{\eta(Y)(\xi f) - Yf\} - \tilde{\mu}\{(hY)f\} \tag{29}$$

From (29) we can write

$$\{(2n+1)\tilde{k} - \frac{r}{2n}\}\{Df - (\xi f)\xi\} + \tilde{\mu}hDf = 0. \tag{30}$$

Operating h in both side of (30) we get

$$\{(2n+1)\tilde{k} - \frac{r}{2n}\}hDf + \tilde{\mu}h^2Df = 0. \tag{31}$$

From which we have

$$hDf = -\frac{\tilde{\mu}h^2Df}{(2n+1)\tilde{k} - \frac{r}{2n}}. \tag{32}$$

Utilizing (13) and (32) in (30) we have

$$\{(2n+1)\tilde{k} - \frac{r}{2n}\}^2(Df - (\xi f)\xi) - \tilde{\mu}^2\tilde{k}\varphi^2 Df = 0. \tag{33}$$

The equation (33) can be written as

$$\{(2n+1)\tilde{k} - \frac{r}{2n})^2 + \tilde{\mu}^2\tilde{k}\}(Df - (\xi f)\xi) = 0. \tag{34}$$

Then either $\{(2n+1)\tilde{k} - \frac{r}{2n})^2 + \tilde{\mu}^2\tilde{k}\} = 0$ or, $(Df - (\xi f)\xi) = 0$.

If $(Df - (\xi f)\xi) = 0$ then

$$Df = (\xi f)\xi. \tag{35}$$

Differentiating covariantly (35) with respect to X we get

$$\nabla_X Df = X(\xi f)\xi + \xi f h'(X). \tag{36}$$

Using (19) and (36) we have

$$fQX = \frac{rf}{2n}X + X(\xi f)\xi + \xi f h'(X). \tag{37}$$

Now from the relation $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$ we get

$$X(\xi f) = \xi(\xi f)\eta(X). \tag{38}$$

and from $\nabla_X Df = f(QX) - \frac{rf}{2n}X$ we have

$$\xi(\xi f) = 2n\tilde{k}f - \frac{rf}{2n}. \tag{39}$$

Using (37), (38) (39) we obtain

$$fQX = \frac{rf}{2n}X + 2n\tilde{k}f - \frac{rf}{2n}\eta(X)\xi + \xi f h'(X). \tag{40}$$

Taking trace of the foregoing equation we obtain

$$f = 0.$$

This completes the proof. □

Now we consider the case $h = 0$. Then from (12) we have

$$R(X,Y)\xi = \tilde{k}(\eta(Y)X - \eta(X)Y), \tag{41}$$

for all smooth vector fields X, Y on M.
Taking inner product with Df in (41) and replacing $X = \xi$ we get

$$g(R(\xi,Y)\xi, Df) = \tilde{k}(\eta(Y)\xi f - Yf). \tag{42}$$

From equation (21) we get

$$R(X,Y)Df = (Xf)QY - (Yf)QX + f\{(\nabla_X Q)Y - (\nabla_Y Q)X\} - \frac{r}{2n}\{(Xf)Y - (Yf)X\} \tag{43}$$

Taking inner product with ξ in both side of (43) we obtain

$$\begin{aligned} g(R(X,Y)Df,\xi) =& 2n\tilde{k}\{(Xf)\eta(Y) - (Yf)\eta(X)\} - \frac{r}{2n}\{(Xf)\eta(Y) - (Yf)\eta(X)\} \\ &+ f\{g((\nabla_X Q)\xi, Y) - g((\nabla_Y Q)\xi, X)\} \end{aligned} \tag{44}$$

Using (19) and (17) in (44) we get

$$g(R(X,Y)Df,\xi) = (2n\tilde{k} - \frac{r}{2n})((Xf)\eta(Y) - (Yf)\eta(X)) - 2fg(X,\varphi Y) \tag{45}$$

Putting $X = \xi$ in the previous equation we obtain

$$g(R(\xi,Y)\xi, Df) = (\frac{r}{2n} - 2n\tilde{k})((\xi f)\eta(Y) - (Yf)) \tag{46}$$

Now (42) and (46) together imply

$$(\frac{r}{2n} - 2n\tilde{k})((\xi f)\eta(Y) - (Yf)) = \tilde{k}(\eta(Y)\xi f - Yf) \tag{47}$$

From the foregoing equation we obtain

$$((\xi f)\eta(Y) - Y(f))\{(2n+1)\tilde{k} - \frac{r}{2n}\} = 0. \tag{48}$$

Suppose that $r \neq 2n(2n+1)\tilde{k}$.
Then from $(\xi f)\eta(Y) - Y(f) = 0$ we have $Df - \xi f\xi = 0$.
Differentiating covariantly $Df - (\xi f)\xi = 0$ with respect to X we get

$$\nabla_X Df = X(\xi f)\xi. \tag{49}$$

Using (19) and (49) we have

$$fQX = \frac{rf}{2n}X + X(\xi f)\xi. \tag{50}$$

Now from the relation $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$ we get

$$X(\xi f) = \xi(\xi f)\eta(X). \tag{51}$$

and from $\nabla_X Df = f(QX) - \frac{rf}{2n}X$ we have

$$\xi(\xi f) = 2n\tilde{k}f - \frac{rf}{2n}. \tag{52}$$

Using (50), (51), (52) we obtain

$$fQX = \frac{rf}{2n}X + 2n\tilde{k}f - \frac{rf}{2n}\eta(X)\xi. \tag{53}$$

Taking trace of the foregoing equation and noting that $\tilde{k} < 0$ we obtain

$$f = 0.$$

Therefore, we are in a position to state the following:

Theorem 3.2. *Let* (M^{2n+1}, g) *be a proper* $(\tilde{k}, \tilde{\mu})$ *almost co-Kähler manifold with* $h = 0$*. If* (g, f) *is a non-trivial solution of the vacuum static equation, then the scalar curvature of* M *is* $2n(2n+1)\tilde{k}$*, which is constant.*

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