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ON STARLIKE FUNCTIONS

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1. Introduction : In a recent paper [1], Ming-Po Chen has considered the class of analytic functions

(1.1) $$f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k, \quad n\geqslant 1$$

satisfying the condition

(1.2) $$|zf'(z)/f(z)-1|<\alpha$$

for a given α, $0<\alpha\leqslant 1$, for $|z|<1$.

It is well known that if S denote the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $|z|<1$, then f is said to be starlike of order α, denoted by $f\in S_\alpha$, if $Re\{zf'(z)/f(z)\}>\alpha$ ($|z|<1$). A subclass $S_{\{\alpha\}}$ of S_α consisting of those $f(z)$ for which $|zf'(z)/f(z)-1|<1-\alpha$ for $|z|<1$ was considered by C. P. Mc Carty [2]. Also it is known that if S denote the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $|z|<1$, then f is said to be convex of order α ($0\leqslant\alpha<1$), denoted by $f\in K_\alpha$, if $Re\{1+zf''(z)/f'(z)\}>\alpha$ ($|z|<1$).

Recently H. Silverman [3] has considered the subclass T (of the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $|z|<1$) consisting of functions expressible in the form

(1.3) $$f(z)=z-\sum_{n=2}^{\infty} |a_n|\, z^n.$$

In this paper we like to consider the class of functions(1.1) of Ming-Po Chen from the view-point of coefficient inequalities. For this purpose we make the following definitions :

Definition 1. If S(n) denote the class of functions (1.1) that are analytic and univalent in the unit disk $|z|<1$, then f is said to be starlike of order α ($0\leqslant\alpha<1$), denoted by $f\in S_\alpha(n)$, if $Re\ \{zf'(z)/f(z)\}>\alpha$ ($|z|<1$).

A subclass $S_{\{\alpha\}}(n)$ of $S_\alpha(n)$ consists of those functions $f(z)$ for which $|zf'(z)/f(z)-1|<1-\alpha$ for $|z|<1$.

Definition 2. If S(n) denote the class of functions (1.1) that are analytic and univalent in the unit disk $|z|<1$, then f is said to be convex of order α ($0\leqslant\alpha<1$), denoted by $f\in K_\alpha(n)$, if $Re\ \{1+zf''(z)/f'(z)\}>\alpha$ ($|z|<1$).

Definition 3. Let T be the subclass of functions (1.1) consisting of functions expressible in the form

(1.4) $$f(z)=z-\sum_{k=n+1}^{\infty} |a_k| z^k,\ n\geqslant 1.$$

Then $T_\alpha(n)$ and $C_\alpha(n)$ are defined respectively as the subclasses of T that are starlike of order α and convex of order α.

2. Sufficient condition for $f(z) \in S_{\{\alpha\}}(n)$

Theorem 1. Let $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k,\ n\geqslant 1.$

If $$\sum_{k=n+1}^{\infty} [(k-\alpha)/(1-\alpha)].\ |a_k| \leqslant 1,$$

then $f(z) \in S_{\{\alpha\}}(n)$ for $\alpha \in [0, 1)$.

Proof. On $|z|=1$, we have

$$(1-\alpha)\ |f(z)| - |zf'(z)-f(z)|$$

$$=(1-\alpha)\ |z+\sum_{k=n+1}^{\infty} a_k z^k| - |\sum_{k=n+1}^{\infty} (k-1) a_k z^k|$$

$$\geqslant(1-\alpha)-\sum_{k=n+1}^{\infty} (k-\alpha)\ |a_k| \geqslant 0, \text{ by hypothesis.}$$

In other words,

$$|zf'(z)/f(z)-1| \leqslant 1-\alpha.$$

Hence by the maximum modulus theorem we have

(2.1) $$|zf'(z)/f(z)-1| < 1-\alpha \text{ for } |z|<1.$$

So $f(z) \in S_{\{\alpha\}}(n)$ for $\alpha \in [0, 1)$.

Special case of theorem 1 was proved by Mc Carty [2] for $n=1$. It may be noted that theorem 1 relates the modulus of coefficients to the order of starlikeness. Further we remark that $f(z)=z-[(1-\alpha)/(k-\alpha)]z^k$ is an extremal function with respect to the above theorem since $|zf'(z)/f(z)-1| = 1-\alpha$ for $z=1$, $\alpha \in [0,1)$, $k=n+1, n+2, \ldots$ and $n\geqslant 1$.

The condition of theorem 1 is not necessary owing to the fact that

$$f(z)=ze^{(1-\alpha)z^n/n} \in S_{\{\alpha\}}(n),$$

whereas

$$\sum_{k=n+1}^{\infty} [(k-\alpha)/(1-\alpha)]\ |a_k| = \sum_{m=1}^{\infty} \frac{mn+1-\alpha}{1-\alpha}\cdot\frac{(1-\alpha)^m}{m!\ n^m}$$

$$>2e^{(1-\alpha)/n}-1>1,$$

for all $\alpha \in [0, 1)$ $n\geqslant 1$.

3. Sufficient condition for $f(z) \in S_\alpha(n)$

Theorem 2. Let $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k,\ n\geqslant 1.$

If $\sum_{k=n+1}^{\infty} [(k-\alpha)/(1-\alpha)].\ |\ a_k\ | \leqslant 1,$

then $f(z) \in S_\alpha(n)$ for $\alpha \in [0, 1)$.

Proof. It is sufficient to show that the values for zf'/f lie in a circle centered at $w=1$ whose radius is $1-\alpha$. In other words, we are to show that $|\ zf'(z)/f(z)-1\ | < 1-\alpha$ for $|\ z\ | < 1$, which is already proved in theorem 1 under the same hypothesis. Hence the theorem is proved.

Special case of theorem 2 has been proved by Silverman [3] for $n=1$. Also particular cases of theorem 2 were proved by Goodman [4] for $n=1, \alpha=0$, and by Schild [5] for $n=1, \alpha=\frac{1}{2}$.

Corollary. Let $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k,\ n\geqslant 1,$

If $\sum_{k=n+1}^{\infty} [k(k-\alpha)/(1-\alpha)]\ |\ a_k\ | \leqslant 1$, then $f(z) \in K_\alpha(n)$.

Proof. $f(z) \in K_\alpha(n)$ if and only if $z f'(z) \in S_\alpha(n)$.

Now since $zf'=z+\sum_{k=n+1}^{\infty} k a_k z^k$, one may replace a_k with ka_k in theorem 2.

4. Necessary and sufficient condition for $f(z) \in T_\alpha(n)$

Theorem 3. A function $f(z)=z-\sum_{k=n+1}^{\infty} |\ a_k\ | z^k,\ n\geqslant 1$, is in $T_\alpha(n)$ if and only if

$$\sum_{k=n+1}^{\infty} [(k-\alpha)/(1-\alpha)]\ |\ a_k\ | \leqslant 1.$$

Proof. The sufficiency part follows from theorem 2. Now to prove the necessary part we assume that

$$(4.1) \quad Re\ \{zf'(z)/f(z)\} = Re \left\{ \frac{z-\sum_{k=n+1}^{\infty} k\ |\ a_k\ |\ z^k}{z-\sum_{k=n+1}^{\infty} |\ a_k\ |\ z^k} \right\} > \alpha, \ (\ |\ z\ | < 1).$$

Choosing values of z on the real axis so that zf'/f is real and letting $z \to 1$ through real values, we obtain from (4·1)

$$1-\sum_{k=n+1}^{\infty} k\ |\ a_k\ | \geqslant \alpha\ (1-\sum_{k=n+1}^{\infty} |\ a_k\ |\ ),$$

which is equivalent to

$$\sum_{k=n+1}^{\infty} [(k-\alpha)/(1-\alpha)]\ |\ a_k\ | \leqslant 1.$$

This completes the proof.

Corollary 1. If $f(z) \in T_\alpha(n)$ then $|a_k| \leqslant (1-\alpha)/(k-\alpha)$, with equality only for functions of the form $f_k(z) = z - [(1-\alpha)/(k-\alpha)]z^k$.

Corollary 2. A function $f(z) = z - \sum_{k=n+1}^{\infty} |a_k| z^k$, $n \geqslant 1$, is in $C_\alpha(n)$ if and only if $\sum_{k=n+1}^{\infty} [k(k-\alpha)/(1-\alpha)] |a_k| \leqslant 1$.

Proof. The proof follows as that of corollary to theorem 2.

5. Remarks on Starlike Functions

We know that [6] an analytic function which is normalized by the condition $f(0) = f'(0) - 1 = 0$, is said to be in the class of functions known as prestarlike of order α, $0 \leqslant \alpha < 1$, if $f * g_\alpha \in S_\alpha$ where $g_\alpha(z) = z/(1-z)^{2(1-\alpha)}$ and $f * g_\alpha$ is the Hadamard product of $f(z)$ and $g_\alpha(z)$. Moreover a necessary and sufficient condition for f to be prestarlike of order α is that the functional

$$G(\alpha, z) = \left\{ f(z) * \frac{g_\alpha(z)}{1-z} \right\} \Big/ \left\{ f(z) * g_\alpha(z) \right\}$$

satisfies $Re\, G(\alpha, z) > 1/2$ $(|z| < 1)$.

Since the Hadamard product of two starlike functions of the same order is a starlike function of the same order, it follows that all starlike functions of order α are obviously prestarlike functions of order α. Hence the necessary condition in order that a function $f(z)$ is starlike of order α is that

$$Re\, G(\alpha, z) > 1/2 \quad (|z| < 1).$$

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