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ON AN ASYMPTOTIC FORMULA FOR A SERIES INVOLVING THE EIGENVALUES OF A DIFFERENTIAL OPERATOR

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1. The Problem : Let $C^k(R)$, $R : 0 \leqslant x < \infty$, be the set of all real-valued functions, having k continuous derivatives on R.

Consider the differential equations

$$(1.1) \qquad \begin{aligned} \frac{d^2u}{dx^2} - pu &= -\lambda u \\ \frac{d^2v}{dx^2} - qv &= -\lambda v \end{aligned}$$

where $p, q \in C^1(R)$ or p, q are absolutely continuous over any compact sub-interval of R, and $\lambda \in C$, the set of all complex numbers. Let u, v be linearly independent, $\phi = \binom{u}{v}$, and $\phi \in D$, the basic Hilbert space $L_2\,[0, \infty)$. Let us assume further that $p\phi$, $q\phi \in D$.

The boundary conditions considered are

$$(1.2) \qquad u(0) = v(0) = 0 \text{ and } u, v \in L_2 \text{ at } \infty$$

$$(1.3) \qquad \text{or} \quad u'(0) = v'(0) = 0 \text{ and } u, v \in L_2 \text{ at } \infty.$$

As usual we call (1.2) the Dirichlet and (1.3) the Neumann boundary conditions.

The differential equations (1.1) with either the Dirichlet or the Neumann boundary conditions, give rise to an eigenvalue problem.

Let $p > 0$, $q > 0$ be steadily increasing in x, for $x \in [0, \infty)$. Then it follows by the analysis of Chakravarty and Sengupta [1], that the sequence of eigenvalues $\{\lambda_n\}$ for the problem (1.1) and (1.2) or the problem (1.1) and (1.3) is positive, and is a discrete collection with $\lim\limits_{n \to \infty} \lambda_n = \infty$.

Let $\psi_n \equiv \binom{\psi_{1n}}{\psi_{2n}}$ be the eigenvector corresponding to the eigenvalue λ_n.

Then $\psi_n \in D$.

The object of the present note is to prove the following theorem.

Theorem.

(i) Let λ_n and ψ_n be the eigenvalue and the eigenvector for the system (1.1), with either the Dirichlet or the Neumann boundary conditions, and let

(ii) $|p(\xi)-p(x)|, |q(\xi)-q(x)| \leqslant C|\xi-x|(p\wedge q)^{\frac{1}{2}}(x)$ for $0<|\xi-x|<1$, C a positive constant and $(p\wedge q)(x) = \text{Min}\,(p(x),\ q(x))$;

(iii) $p(\xi),\ q(\xi) \leqslant K_0 \exp\,[\frac{1}{2}|\xi-x|\ (p\wedge q)^{\frac{1}{2}}(x)]$ for $|\xi-x|>1$, K_0, a positive constant;

(iv) $\int_0^\infty p^{-\frac{1}{2}}\,dx$ and $\int_0^\infty q^{-\frac{1}{2}}\,dx$, are convergent.

(v) $(p\wedge q)(x) \geqslant \frac{1}{4}x^2\,(p\wedge q)(a)$ for all sufficiently large x; $0<a<\infty$;

(vi) $|p(x)-q(x)| \leqslant A\,e^{-\bar{a}_0 x}$, $A, \bar{a}_0$ positive constants.

Then (i) $\sum_{n=0}^{\infty} \frac{1}{\lambda_n^2}$ is convergent;

(ii) as $\mu\to\infty$, $\sum_{n=0}^{\infty} \frac{1}{(\lambda_n+\mu)^2} \sim \frac{1}{4}\,I$, where

$$I=\int_0^\infty \left[\frac{1}{(\mu+p)^{\frac{3}{2}}}+\frac{1}{(\mu+q)^{\frac{3}{2}}}\right] dx.$$

2. Proof of the theorem.

If $g(\xi, x, K) \equiv \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22}\end{pmatrix}$ be the Green's matrix for the "Fourier system", i.e. the system

$$\frac{d^2u}{d\xi^2} = K^2\,u(\xi)$$

(2.1)

$$\frac{d^2v}{d\xi^2} = K^2\,v(\xi)$$

with either the Dirichlet or the Neumann boundary conditions, then it is easy to derive, that for any $x \in [0, \infty)$, and with $K^2 = \mu + p(x)$, where $\mu>0$,

$$\frac{\psi_{1n}(x)}{\lambda_n+\mu}=-\int_0^\infty \left(g_{11}(\xi,x,K),\ g_{12}(\xi,x,K)\right)\begin{pmatrix}\psi_{1n}(\xi)\\ \psi_{2n}(\xi)\end{pmatrix}d\xi$$

$$+\frac{1}{\lambda_n+\mu}\int_0^\infty \left(g_{11}(\xi,x,K)\ \ g_{12}(\xi,x,K)\right)\begin{pmatrix}p(\xi)-p(x) & 0\\ 0 & q(\xi)-q(x)\end{pmatrix}\psi_n(\xi)d\xi$$

(2.2) $\qquad = a_n(x)+b_n(x)$, say.

And

$$\frac{\psi_{2n}(x)}{\lambda_n+\mu}=-\int_0^\infty \left(g_{21}(\xi,x,K)\ \ g_{22}(\xi,x,K)\right)\begin{pmatrix}\psi_{1n}(\xi)\\ \psi_{2n}(\xi)\end{pmatrix}d\xi$$

$$+\frac{1}{\lambda_n+\mu}\int_0^\infty \left(g_{21}(\xi,x,K)\ \ g_{22}(\xi,x,K)\right)\begin{pmatrix}p(\xi)-p(x) & 0\\ 0 & q(\xi)-q(x)\end{pmatrix}\psi_n(\xi)d\xi$$

$$+\frac{1}{\lambda_n+\mu}\int_0^\infty \left(g_{21}(\xi,x,K)\ \ g_{22}(\xi,x,K)\right)\begin{pmatrix}0 & 0\\ 0 & q(x)-p(x)\end{pmatrix}\psi_n(\xi)d\xi$$

(2.2)^A $\qquad =\alpha_n(x)+\beta_n(x)+\gamma_n(x)$, say

It can be easily verified, by the application of the Schwarz inequality, that the infinite integrals involved are convergent.

It is easy to see that the explicit form of the Green's matrix for (2.1), with Dirichlet's and Neumann's boundary conditions are, respectively,

(i) with Dirichlet's boundary conditions,

$$(g_{ij}(\xi,z,K))=M(\xi,z)\,E_2 \quad \text{for } z\leqslant\xi$$
$$=M(z,\xi)\,E_2 \quad \text{for } z>\xi$$

where $M(\xi,z)=\dfrac{e^{-K(z+\xi)}-e^{-K(\xi-z)}}{2K}$, and E_2 is the unit matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

(ii) with Neumann's boundary conditions,

$$(g_{ij}(\xi,z,K))=L(\xi,z)\,E_2 \quad \text{for } z\leqslant\xi$$
$$=L(z,\xi)\,E_2 \quad \text{for } z>\xi,$$

where $L(\xi,z)=\dfrac{e^{-K(z+\xi)}+e^{-K(\xi-z)}}{-2K}$.

Therefore

$$\text{(2.3)}\qquad \int_0^\infty g_{11}{}^2(\xi,x,K)\,d\xi=\begin{cases}\dfrac{1}{4K^3}-\dfrac{e^{-2Kx}}{4K^3}-\dfrac{xe^{-2Kx}}{2K^2}, & \text{with Dirichlet's boundary conditions;}\\[2ex] \dfrac{1}{4K^3}+\dfrac{e^{-2Kx}}{4K^3}+\dfrac{xe^{-2Kx}}{2K^2}, & \text{with the Neumann boundary conditions.}\end{cases}$$

In the following we prove the theorem for the system (1.1), with Dirichlet's boundary conditions. The same analysis holds, when we take the system with Neumann's boundary conditions.

It follows from (2.2) and (2.3), with $K^2=\mu+p$, and the Parseval relation, that

(2.4) $$\int_0^\infty \sum_{n=0}^\infty a_n^2(x)\,dx = \frac{1}{4}\int_0^\infty \frac{dx}{(\mu+p)^{\frac{3}{2}}} + O\left(\mu^{-1}\int_0^\infty \frac{dx}{p^{\frac{1}{2}}}\right)$$ as μ tends to infinity.

A similar result holds for $\alpha_n(x)$ in (2.2)^A.

From (2.2)

$$b_n(x) = \frac{1}{\lambda_n+\mu}\Bigg[\int_{R_1}(g_{11}, g_{12})\begin{pmatrix} p(\xi)-p(x) & 0 \\ 0 & q(\xi)-q(x)\end{pmatrix}\psi_n(\xi)d\xi$$

$$+\int_{R_2}(g_{11}, g_{12})\begin{pmatrix} p(\xi) & 0 \\ 0 & q(\xi)\end{pmatrix}\psi_n(\xi)d\xi - \int_{R_2}(g_{11}, g_{12})\begin{pmatrix} p(x) & 0 \\ 0 & q(x)\end{pmatrix}\psi_n(\xi)d\xi\Bigg]$$

(2.5) $$=\frac{1}{\lambda_n+\mu}\left[\, b_{1n}(x) + b_{2n}(x) - b_{3n}(x)\right], \text{ say,}$$

where $R_1 \equiv |\xi-x| \leqslant 1 \subset R$ and $R_2=R\setminus R_1$. Using the conditions (ii), satisfied by $|p(\xi)-p(x)|$, $|q(\xi)-q(x)|$, holding for R_1, it follows that

$$b_{1n}^2 \leqslant \frac{4c^2}{\mu p^{\frac{1}{2}}(x)}, \text{ leading to}$$

(2.6) $$\int_0^\infty b_{1n}^2(x)\,dx = O\left(\mu^{-1}\int_0^\infty \frac{dx}{p^{\frac{1}{2}}}\right), \text{ as } \mu\to\infty.$$

Similarly, using the conditions (iii) satisfied by $p(\xi)$, $q(\xi)$ holding for R_2,

(2.7) $$\int_0^\infty b_{2n}^2(x)\,dx = O(\mu^{-1}), \text{ as } \mu\to\infty.$$

Again $$b_{3n}^2 < p^2(x)\int_{R_2} g_{11}^2\,d\xi \int_{R_2}\psi_m^2\,d\xi < \frac{p^2\exp(-2(\mu+p)^{\frac{1}{2}})}{(\mu+p)^{\frac{3}{2}}},$$

which leads to

(2.8) $$\int_0^\infty b_{3n}^2(x)\,dx = O(\mu^{-1}) \text{ as } \mu\to\infty.$$

Hence, altogether, we have from (2.5)

$$\int_0^\infty b_n^2(x)dx = O\left(\frac{\mu^{-1}}{(\lambda_n+\mu)^2}\right), \quad \text{as } \mu\to\infty. \tag{2.9}$$

A similar result holds also for $\beta_n(x)$.

To find $\int_0^\infty \gamma_n^2(x)\, dx$, we have

$$\gamma_n^2(x) = \frac{(q(x)-p(x))^2}{(\lambda_n+\mu)^2}\left(\int_0^\infty g_{22}(\xi, x, \mathrm{K})\, \psi_{2n}(\xi)\, d\xi\right)^2$$

$$\leqslant \frac{(q(x)-p(x))^2}{(\lambda_n+\mu)^2}\int_0^\infty g_{22}^2(\xi, x, \mathrm{K})\, d\xi.$$

We have therefore,

$$\int_0^\infty \gamma_n^2(x)\, dx = O\left(\mu^{-1}\frac{1}{(\lambda_n+\mu)^2}\right), \text{ by condition (vi).}$$

Therefore from (2.2) and (2.2)^A, for any positive integer m, we obtain by using the condition (v)

$$\mathrm{S}_m \leqslant \tfrac{1}{2}\int_0^\infty \frac{dx}{(\mu+p)^{3/2}} + O(\mu^{-1}\,\mathrm{S}_m) + O(\mu^{-1}), \quad \text{as } \mu\to\infty, \text{ where} \tag{2.10}$$

$$\mathrm{S}_m = \sum_{n=0}^{m}\frac{1}{(\lambda_n+\mu)^2}.$$

Hence $\sum_{n=0}^{\infty}\frac{1}{\lambda_n^2}$ is convergent, when $\int_0^\infty p^{-\frac{1}{2}}dx$ is so.

Putting $\mathrm{K}^2=\mu+q$, and proceeding as before, we also obtain that

$$\sum_{n=0}^{\infty}\frac{1}{\lambda_n^2} \text{ is convergent, when } \int_0^\infty q^{-\frac{1}{2}}(x)\, dx \text{ is so.}$$

Since $\int_0^\infty \sum_{n=0}^{m}\left[a_n(x)\, b_n(x)\right] dx = O\left[\, \mathrm{I}_1^{\frac{1}{2}}\, \mathrm{S}^{\frac{1}{2}}\, \mu^{-\frac{1}{2}}\,\right]$ where

$\mathrm{I}_1 = \int_0^\infty \frac{dx}{(\mu+p)^{3/2}}$ and $\mathrm{S}=\sum_{n=0}^{\infty}\frac{1}{(\lambda_n+\mu)^2}$, it follows by squaring (2.2), proceeding as before and making m tend to infinity, that

$$S\int_0^\infty \psi_{1n}^2\,dx = \tfrac{1}{4} I_1 + O(\mu^{-1}S) + O(\mu^{-1}) + O(\mu^{-\frac{1}{2}} S^{\frac{1}{2}} I_1^{\frac{1}{2}}),$$

as $\mu\to\infty$, with a similar relation for $S\int_0^\infty \psi_{2n}^2\,dx$.

Therefore, by addition

(2.11) $S=\tfrac{1}{2} I_1+O(\mu^{-1}S)+O(\mu^{-1})+O(\mu^{-\frac{1}{2}} S^{\frac{1}{2}} I_1^{\frac{1}{2}})$, as $\mu\to\infty$.

Again, proceeding similarly with $K^2=\mu+q$, we obtain

(2.12) $S=\tfrac{1}{2} I_2+O(\mu^{-1}S)+O(\mu^{-1})+O(\mu^{-\frac{1}{2}} S^{\frac{1}{2}} I_2^{\frac{1}{2}})$ as $\mu\to\infty$,

where $I_2=\int_0^\infty \frac{dx}{(\mu+q)^{3/2}}$.

Combining (2.11) and (2.12), we obtain

$S\sim\tfrac{1}{2} I$, as $\mu\to\infty$.

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