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ON THE PRODUCT OF FUZZY SUBSETS

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1. Introduction.

Since the introduction of the concept of fuzzy subsets by Zadeh [8] in 1965, many researchers have contributed to the development of the theory in various directions and the theory has been found to be extremely fruitful in the application field ([1], [2], [3], [4], [6], [7], [9]). We know that if U be the reference set, any ordinary subset A of U can be represented by the corresponding characteristic function f_A mapping the elements of U to the set {0, 1}. If instead, a function $f_{\underset{\sim}{A}}$ maps U to the closed interval [0, 1], the parallel notion is called a fuzzy subset of U and will be denoted by $\underset{\sim}{A}$. Equality, inclusion, union, intersection and complementation are defined as below

Def. (i) $\underset{\sim}{A}=\underset{\sim}{B} \rightleftharpoons f_{\underset{\sim}{A}}(x)=f_{\underset{\sim}{B}}(x)$ for all $x \in U$

(ii) $\underset{\sim}{A}\subseteq\underset{\sim}{B} \rightleftharpoons f_{\underset{\sim}{A}}(x)\leqslant f_{\underset{\sim}{B}}(x)$ for all $x \in U$

(iii) $\underset{\sim}{A}\cup\underset{\sim}{B} \rightleftharpoons \max\,[f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{B}}(x)]$ for all $x \in U$

(iv) $\underset{\sim}{A}\cap\underset{\sim}{B} \rightleftharpoons \min\,[f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{B}}(x)]$ for all $x \in U$

(v) $\bar{\underset{\sim}{A}} \rightleftharpoons 1-f_{\underset{\sim}{A}}(x)$ for all $x \in U$.

The notion of fuzzy subset and the above definitions obviously generalise the theory of ordinary subsets of a set.

The notion of fuzzy relation among members of an ordinary set and some immediate consequences have been brilliantly studied by Kaufmann [5] and some further developments in this respect have been made in [2]. A fuzzy relation in an ordinary set S is a fuzzy subset of the product S×S. The product of fuzzy subsets has, however, not been defined so far. In this paper we shall introduce this notion and shall derive some theorems showing thereby analogies and departures from ordinary set theory.

2. Product of fuzzy subsets.

Definition. Let U be the reference set and [0, 1] the membership set. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be fuzzy subsets of U defined by the membership functions $f_{\underset{\sim}{A}}$ and $f_{\underset{\sim}{B}}$ respectively. The product $\underset{\sim}{A}\times\underset{\sim}{B}$ is the fuzzy subset of U×U defined by the membership function

$$f_{\underset{\sim}{A}\times\underset{\sim}{B}} : f_{\underset{\sim}{A}\times\underset{\sim}{B}}(x,y)=\min\,[f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{B}}(y)] \text{ for all } x,y \in U.$$

Example : Let $U=\{a, b, c, d\}$,

A : $f_A(a)=\cdot 1$, $f_A(b)=\cdot 2$, $f_A(c)=0$, $f_A(d)=1$,

B : $f_B(a)=0$, $f_B(b)=1$, $f_B(c)=1$, $f_B(d)=\cdot 5$.

Then $A \times B$ is the fuzzy subset of $U \times U$ defined by the membership function $f_{A \times B}$ which maps

$(a,a) \to 0$	$(b,a) \to 0$	$(c,a) \to 0$	$(d,a) \to 0$
$(a,b) \to \cdot 1$	$(b,b) \to \cdot 2$	$(c,b) \to 0$	$(d,b) \to 1$
$(a,c) \to \cdot 1$	$(b,c) \to \cdot 2$	$(c,c) \to 0$	$(d,c) \to 1$
$(a,d) \to \cdot 1$	$(b,d) \to \cdot 2$	$(c,d \to 0$	$(d,d) \to \cdot 5$.

This is a straightway generalisation of the cartesian product of two ordinary subsets of U.

The following results follow directly from the definition.

Theorem 2.1. (i) $A \times B \neq B \times A$

(ii) $(A \times B) \times C = A \times (B \times C)$

(iii) $A \times \phi = \phi \times A = \phi$, where ϕ is the null fuzzy set.

(iv) $f_{A \times U}(x,y) = f_A(x)$ for all pairs x,y of U.

We give below some more results that hold also in case of ordinary set theory.

Theorem 2.2. (i) $A \subseteq B \Rightarrow A \times C \subseteq B \times C$

and $C \times A \subseteq C \times B$

(ii) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

(iii) $(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$

(iv) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

$C \times (A \cap B) = (C \times A) \cap (C \times B)$

(v) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$(B \cup C) \times A = (B \times A) \cup (C \times A)$.

Proof. (i) $A \subseteq B$ implies $f_A(x) \leqslant f_B(x)$ for all x in U.

Now, $f_{A \times C}(x,y) = \min [f_A(x), f_C(y)]$
and $f_{B \times C}(x,y) = \min [f_B(x), f_C(y)]$ } for all $x,y \in U$.

Case I. $f_A(x) \leqslant f_C(y)$.

Then $f_A(x) \leqslant f_C(y)$ and $f_B(x)$.

Hence $f_A(x) \leqslant \min[f_B(x), f_C(y)]$.

Hence $\min[f_A(x), f_C(y)] = f_A(x) \leqslant \min[f_B(x), f_C(y)]$.

Case II. $f_C(y) < f_A(x) \leqslant f_B$.

Obviously, $\min[f_A(x), f_C(y)] = \min[f_B(x), f_C(y)]$.

So, $A \times C \subseteq B \times C$.

Similarly, $C \times A \subseteq C \times B$.

(ii) $f_{(A \cap B) \times (C \cap D)}(x, y) = \min[\min(f_A(x), f_B(x)), \min(f_C(y), f_D(y))]$

$= \min[f_A(x), f_B(x), f_C(y), f_D(y)]$

$= \min[\min(f_A(x), f_C(y)), \min(f_B(x), f_C(y))]$

$= f_{(A \times C) \cap (B \times D)}(x, y)$.

(iii) $f_{(A \cup B) \times (C \cup D)}(x, y) = \min[\max(f_A(x), f_B(x)), \max(f_C(y), f_D(y))]$.

$f_{(A \times C) \cup (B \times D)}(x, y) = \max[\min(f_A(x), f_C(y)), \min(f_A(x), f_D(y))]$.

Taking $f_A(x) = 1$, $f_B(x) = 0$, $f_C(y) = 0$, $f_D(y) = 1$, it is clear that the two membership functions are not same.

(iv) $(A \cap B) \times C = (A \cap B) \times (C \cap C) = (A \times C) \cap (B \times C)$ by (ii).

Similarly, $C \times (A \cap B) = (C \times A) \cap (C \times B)$.

(v) $f_{A \times (B \cup C)}(x, y) = \min[f_A(x), \max(f_B(y), f_C(y))]$.

$f_{(A \times B) \cup (A \times C)}(x, y) = \max[\min(f_A(x), f_B(y)), \min(f_A(x), f_C(y))]$.

There are six possibilies

$f_A(x) \geqslant f_B(y) \geqslant f_C(y)$

$f_A(x) \geqslant f_C(y) \geqslant f_B(y)$

$f_B(y) \geqslant f_A(x) \geqslant f_C(y)$

$f_B(y) \geqslant f_C(y) \geqslant f_A(x)$

$f_C(y) \geqslant f_A(x) \geqslant f_B(y)$

$f_C(y) \geqslant f_B(y) \geqslant f_A(x)$.

In all the cases the above membership functions are identical.

Definition. The difference, disjunctive sum, algebraic product and algebraic sum of two fuzzy sets have been defined respectively as follows [5].

$A - B = A \cap \bar{B}.$

$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B).$

$A \, . \, B$ by the membership function $f_A(x) . f_B(x)$.

$A \hat{+} B$ by $f_A(x) + f_B(x) - f_A(x) . f_B(x)$.

The following theorem contains the departure from the theory of ordinary sets.

Theorem 2.2. (i) $(A - B) \times C \not\equiv (A \times C) - (B \times C)$.

(ii) $(A \oplus B) \times C \not\equiv (A \times C) \oplus (B \times C)$.

(iii) $(A \, . \, B) \times C \not\equiv (A \, . \, C) \times (B \, . \, C)$.

(iv) $(A \hat{+} B) \times C \not\equiv (A \times C) \hat{+} (B \times C)$.

Proof : (1) $f_{(A-B)\times C}(x, y) = \min [f_{A \cap \bar{B}}(x), f_C(y)]$

$= \min [f_A(x), 1 - f_B(x), f_C(y)].$

and $f_{(A\times C)-(B\times C)}(x,y) = \min [\min (f_A(x), f_C(y)), 1 - \min (f_B(x), f_C(y))].$

Taking $f_A(x) = {\cdot}5$, $f_B(x) = 1$, $f_C(y) = {\cdot}5$ the non-identity of the above membership functions are established.

(ii) $(A \oplus B) \times C = [(A \cap \bar{B}) \cup (\bar{A} \cap B)] \times C$

$= [(A \cap \bar{B}) \times C] \cup [(\bar{A} \cap B) \times C]$ by th. 2·2

$= [(A \times C) \cap (\overline{B \times C})] \cup [(\bar{A} \times C) \cap (B \times C)]$

by th. 2·2

So $f_{(A \oplus B)\times C}(x, y) =$

$\max [\min (f_A(x), 1 - f_B(x), f_C(y)), \min (1 - f_A(x), f_B(x), f_C(y))],$

and $(A \times C) \oplus (B \times C) =$

$[(A \times C) \cap (\overline{B \times C})] \cup [(\overline{A \times C}) \cap (B \times C)].$

So $f_{(A \times C) \oplus (B \times C)}(x, y) =$

$\max [\min \{\min (f_A(x), f_C(y)), 1 - \min (f_B(x), f_C(y))\},$

$\min \{1 - \min (f_A(x), f_C(y)), \min (f_B(x), f_C(y))\}].$

The non-identity of the two functions is proved by taking

$$f_{\underset{\sim}{A}}(x) = \cdot 9,\ f_{\underset{\sim}{B}}(x) = \cdot 8,\ f_{\underset{\sim}{C}}(y) = \cdot 3.$$

(iii) $f_{(\underset{\sim}{A}.\underset{\sim}{B})\times\underset{\sim}{C}}(x, y) = \min [f_{\underset{\sim}{A}}(x).f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y)].$

$f_{(\underset{\sim}{A}\times\underset{\sim}{C}).(\underset{\sim}{B}\times\underset{\sim}{C})}(x, y) = \min [f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{C}}(y)].\min [f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y)].$

Take $f_{\underset{\sim}{A}}(x) = \cdot 5,\ f_{\underset{\sim}{B}}(x) = \cdot 5,\ f_{\underset{\sim}{C}}(y) = \cdot 1.$

(iv) $f_{(\underset{\sim}{A}\hat{+}\underset{\sim}{B})\times\underset{\sim}{C}}(x, y) = \min [f_{\underset{\sim}{A}}(x) + f_{\underset{\sim}{B}}(x) - f_{\underset{\sim}{A}}(x).f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y)].$

$f_{(\underset{\sim}{A}\times\underset{\sim}{C})\hat{+}(\underset{\sim}{B}\times\underset{\sim}{C})}(x, y) = \min [f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{C}}(y)] + \min [f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y)]$

$- \min [f_{\underset{\sim}{A}}(x), f_{\underset{\sim}{C}}(y)].\min [f_{\underset{\sim}{B}}(x), f_{\underset{\sim}{C}}(y)].$

Take $f_{\underset{\sim}{A}}(x) = 1,\ f_{\underset{\sim}{B}}(x) = 1,\ f_{\underset{\sim}{C}}(y) = \cdot 5.$

Remarks. In ordinary set theory equality holds in all the cases stated in theorem 2.3. Algebraic product and sum reduce to intersection and union when the sets are ordinary.

3. Conclusion.

Fuzzy relation between two ordinary sets has been extensively studied ([5], [2]). With the introduction of the product of two fuzzy subsets, a natural possibility has developed to think about the concept of relation between two fuzzy subsets that can be defined as a fuzzy subset of the product. Reflexivity, symmetry etc. may then be defined and corresponding results obtained in this direction will be presented in a future publication.

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