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ON A PAIR OF GENERATING RELATIONS FOR SOME SPECIAL FUNCTIONS FROM THE VIEW OF LIE-ALGEBRA

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1. Introduction : Starting from the infinitesimal operators R and L, the elements of Lie-algebra for a particular special function, which raise and lower the indices of the special function, we can generate the finite operatore (exp a R), (exp b L) of the corresponding Lie-group. Now since any element of the said Lie-group operates on the function in the following ways :

(i) it shifts the argument of the function

(ii) it produces an infinite sum of functions with unchanged arguments but with shifted indices,

the desired generating function can be obtained by equating the two results.

The composition law

$$(\exp aR)\,(\exp bL) = \exp(aR + bL)$$

will hold or not according as R, L commute or not. In case when $[R, L] \neq 0$, we shall operate (exp aR) (exp bL) and (exp bL) (exp aR) successively on the function concerned in order to derive a pair of generating relations for the function. This method was already suggested by S. K. Chatterjea [1].

The object of the present paper is to follow the method of Chatterjea in order to derive a pair of generating relations for the Laguerre and Bessel polinomials seperately from a different view point.

2. Laguerre polynomials : From the second order differential equation

$$xD^2y + (1+\alpha-x)\,Dy + ny = 0, \quad (D \equiv d/dx)$$

for the Laguerre polynomials, we notice that

$$(2.1) \quad R\,[L_n^{(\alpha)}(x)\,y^{\alpha}] = -L_n^{(\alpha+1)}(x)\,y^{\alpha+1}$$

$$L\,[L_n^{(\alpha)}(x)\,y^{\alpha}] = (n+\alpha)\,L_n^{(\alpha-1)}(x)\,y^{\alpha-1},$$

where

$$(2.2) \quad R = y\frac{\partial}{\partial x} - y, \quad L = xy^{-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

and $[R, L] = 1$ i.e., $[R, L] \neq 0$,

We shall now apply the operator (exp aR) (exp bL) to ($L_n^{(\alpha)}(x)\, y^{\alpha}$).

We have

(2.3) $(\exp a\mathrm{R})\, f(x, y) = f(x + ay, y)$

(2.4) $(\exp b\mathrm{L})\, f(x, y) = e^{-ay} f\left(\dfrac{x\,(y+b)}{y},\ y+b\right).$

Thus we get

$$(\exp a\mathrm{R})\,(\exp b\mathrm{L})\,(L_n^{(\alpha)}(x)\, y^{\alpha}) = e^{-ay}\,(y+b)^{\alpha}\, L_n^{(\alpha)}\left(\frac{(x+ay)\,(y+b)}{y}\right).$$

On the other hand,

(exp aR) (exp bL) ($L_n^{(\alpha)}(x)\, y^{\alpha}$)

$$= \sum_{m=0}^{\infty} \frac{(-ay)^m}{m\,!} \sum_{p=0}^{\infty} \frac{b^p}{p\,!}\,(n+\alpha-p+1)_p\, y^{\alpha-p}\, L_n^{(\alpha-p+m)}(x).$$

Equating the above two results we get

$$(2.5)\quad \sum_{m=0}^{\infty} \frac{(-ay)^m}{m\,!} \sum_{p=0}^{\infty} \frac{b^p}{p\,!}\,(n+\alpha-p+1)_p\, y^{\alpha-p}\, L_n^{(\alpha-p+m)}(x) = e^{-ay}\,(b+y)^{\alpha}\, L_n^{(\alpha)}\left(\frac{(x+ay)\,(y+b)}{y}\right).$$

Next we shall apply the operator (exp bL) (exp aR) to ($L_n^{(\alpha)}(x)\, y^{\alpha}$).

First we observe that

$$(\exp b\mathrm{L})\,(\exp a\mathrm{R})\,(L_n^{(\alpha)}(x)\, y^{\alpha}) = e^{-a(y+b)}\,(y+b)^{\alpha}\, L_n^{(\alpha)}\left(\frac{(x+ay)\,(y+b)}{y}\right).$$

On the other hand,

(exp bL) (exp aR) ($L_n^{(\alpha)}(x)\, y^{\alpha}$)

$$= \sum_{m=0}^{\infty} \frac{(-ay)^m}{m\,!} \sum_{p=0}^{\infty} \frac{b^p}{p\,!}\,(n+\alpha+m-p+1)_p\, y^{\alpha-p}\, L_n^{(\alpha+m-p)}(x).$$

Equiting the above two results we get

$$(2.6)\quad \sum_{m=0}^{\infty} \frac{(-ay)^m}{m\,!} \sum_{p=0}^{\infty} \frac{b^p}{p\,!}\,(n+\alpha+m-p+1)_p\, y^{\alpha-p}\, L_n^{(\alpha+m-p)}(x) = e^{-a(y+b)}\,(y+b)^{\alpha}\, L_n^{(\alpha)}\left(\frac{(x+ay)\,(y+b)}{y}\right).$$

It is interesting to remark that in particular when $b=0$, both the relations (2.5) and (2.6) reduce to the well-known generating relation [2, p 373] :

$$(2.7) \quad \sum_{m=0}^{\infty} \frac{y^m}{m!} L_n^{(\alpha+m)}(x) = e^y L_n^{(\alpha)}(x-y).$$

Thus the pair of generating relations (2.5) and (2.6) can be considered as the extension of (2.7).

Bessel Polynomials : From the second order differential equation :

$$x^2 D^2 y + (\alpha x + \beta) Dy - n(n+\alpha-1) y = 0$$

for the Bessel polynomials, we notice that [3] :

$$(3.1) \quad R = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (n-1) y, \quad L = \frac{x^2}{y} \frac{\partial}{\partial x} - \frac{nx-\beta}{y}$$

and $[R, L] = -\beta$ i.e. $[R, L] \neq 0$,

such that,

$$(3.2) \quad R\,(Y_n^{(\alpha)}(x)\, y^{\alpha}) = (n+\alpha-1)\, Y_n^{(\alpha+1)}(x)\, y^{\alpha+1},$$

$$L\,(Y_n^{(\alpha)}(x)\, y^{\alpha}) = \beta Y_n^{(\alpha-1)}(x)\, y^{\alpha-1}.$$

We shall apply the operator $(\exp aR)$ $(\exp bL)$ to $(Y_n^{(\alpha)}(x)\, y^{\alpha})$.

We have

$$(3.3) \quad (\exp aR) f(x, y) = (1-ay)^{-n+1} f(x/(1-ay), y/(1-ay))$$

$$(3.4) \quad (\exp bL) f(x, y) = \left(1 - b\frac{x}{y}\right)^n e^{b\beta/y} f\left(\frac{xy}{y-bx}, y\right).$$

Thus we get,

$(\exp aR)$ $(\exp bL)$ $(Y_n^{(\alpha)}(x)\, y^{\alpha})$

$$= (1-ay)^{1-\alpha-n} (y-bx)^n y^{\alpha-n} e^{b\beta(1-ay)/y}\, Y_n^{(\alpha)}\left(\frac{xy}{(y-bx)(1-ay)}\right).$$

On the other hand

$(\exp aR)$ $(\exp bL)$ $(Y_n^{(\alpha)}(x)\, y^{\alpha})$

$$= \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{(b\beta)^p}{p!} \frac{a^m}{m!} (n+\alpha-p-1)_m\, Y_n^{(\alpha-p+m)}(x)\, y^{\alpha-p+m}.$$

Equating the above two results we get,

(3.5) $(1-ay)^{1-\alpha-n}\,(y-bx)^n\,e^{b\beta(1-ay)/y}\,Y_n^{(\alpha)}\,(xy/(y-bx)\,(1-ay))$

$$=\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{(b\beta)^p}{p\,!}\,\frac{a^m}{m\,!}\,(n+\alpha-p-1)_m\,Y_n^{(\alpha-p+m)}\,(x)\,y^{\alpha-p+m}.$$

Now we shall apply the operator (exp bL) (exp aR) to ($Y_n^{(\alpha)}$ (x) y^α).

First we observe that

(exp bL) (exp aR) ($Y_n^{(\alpha)}$ (x) y^α)

$$=(1-ay)^{1-\alpha-n}\,(y-bx)^n\,y^{\alpha-n}\,e^{b\beta/y}\,Y_n^{(\alpha)}\,(xy/(1-ay)\,(y-bx))$$

On the other hand

(exp bL) (exp aR) ($Y_n^{(\alpha)}$ (x) y^α)

$$=\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{a^m}{m\,!}\,\frac{(b\beta)^p}{p\,!}\,(n+\alpha-1)_m\,Y_n^{(\alpha+m-p)}\,(x)\,y^{\alpha+m-p}.$$

Equating the above two results we get,

(3.6) $(y-bx)^n\,(1-ay)^{1-\alpha-n}\,e^{b\beta/y}\,Y_n^{(\alpha)}\,(xy/(1-ay)\,(y-bx))$

$$=\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{a^m}{m\,!}\,\frac{(b\beta)^p}{p\,!}\,(n+\alpha-1)_m\,Y_n^{(\alpha+m-p)}\,(x)\,y^{m+n-p}.$$

Notice that in particular when $b=0$, both the relations (3.5) and (3.6) reduce to the well known generating relation [4, p 50]

(3.7) $$(1-y)^{1-\alpha-n}\,Y_n^{(\alpha)}\left(\frac{x}{1-y}\right)=\sum_{m=0}^{\infty}\frac{(n+\alpha-1)_m}{m\,!}\,Y_n^{(\alpha+m)}\,(x)\,y^m.$$

Thus the pair of generating relations (3.5) and (3.5) can be considered as the extension of (3.7)

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