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ON PARTIAL DIFFERENTIAL OPERATORS FOR $F(-n, \beta ; \gamma ; x)$

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1. Introduction : In the application of Lie algebra to a special function it is usual to find two operators (called the generators of Lie algebra) which raise and lower the index (or parameter) of the special function under consideration [1]. The object of this paper is to present two partial differential operators, viz.

$$A = x(1-x)\, yu^{-1} \frac{\partial}{\partial x} - xyzu^{-1} \frac{\partial}{\partial z} + y \frac{\partial}{\partial u} - yu^{-1}$$

$$B = x(1-x)\, yz^{-1} u^{-1} \frac{\partial}{\partial x} + xy^2 z^{-1} u^{-1} \frac{\partial}{\partial y} - xyu^{-1} \frac{\partial}{\partial z} + yz^{-1} \frac{\partial}{\partial u} - (1-x)\, yz^{-1} u^{-1},$$

such that A raises the index n and at the same time lowers the parameter γ of $F(-n, \beta ; \gamma ; x)$, while B raises the index and lowers both parameters β, γ of $F(-n, \beta ; \gamma ; x)$ at the same time.

In other words, we have,

$$(1.1) \quad A\,[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)] = (\gamma-1)\, y^{n+1} zu^{\gamma-1} F(-n-1, \beta ; \gamma-1 ; x)$$

$$B\,[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)] = (\gamma-1)\, y^{n+1} z^{\beta-1} u^{\gamma-1} F(-n-1, \beta-1 ; \gamma-1 ; x).$$

The extended forms of the transformation groups generated by the operators A, B are given by

$$(1.2) \quad \exp(aA)\, f(x, y, z, u) = \frac{u}{u+ay} f\left(x \frac{u+ay}{u+axy},\ y,\ \frac{zu}{u+axy},\ u+ay\right)$$

$$\exp(bB)\, f(x, y, z, u) = \frac{zu}{zu+by(1-x)} f\left(x \frac{zu+by(1-x)}{zu},\ \frac{yzu}{zu-bxy},\ \frac{zu-bxy}{u},\ u\, \frac{zu+by(1-x)}{zu-bxy}\right).$$

The introduction of such operators for $F(-n, \beta ; \gamma ; x)$ helps us to derive the following generating relations :

$$(1.3) \quad (1-t)^{\gamma-1} (1-xt)^{-\beta} F\left(-n, \beta ; \gamma ; x \frac{1-t}{1-xt}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-\gamma+1)_m}{m!} F(-n-m, \beta ; \gamma-m ; x)\, t^m,$$

where $|t| < \min(1, |x|^{-1})$.

(1.4) $$(1+xt)^{\beta-n-\nu}(1+t(x-1))^{\gamma-1} F(-n, \beta; \gamma; x+xt(x-1))$$

$$= \sum_{m=0}^{\infty} \frac{(-\gamma+1)_m}{m!} F(-n-m, \beta-m; \gamma-m; x)\, t^m$$

where $x \neq 1$, $|t| < \min(|x|^{-1}, |1-x|^{-1}, |x|^{-1}|1-x|^{-1})$

Furthermore, we have proved the following general theorems on generating relations for hypergeometric polynomials $F(-n, \beta; \gamma; x)$.

Theorem I : If there exists a generating function of the form :

$$F(x, t) = \sum_{n=0}^{\infty} a_n F(-n, \beta; \gamma-n; x)\, t^n.$$

Then,

(1.5) $$\sum_{n=0}^{\infty} F(-n, \beta; \gamma-n; x)\, \sigma_n(y)\, t^n.$$

$$= (1-t)^{\gamma-1}(1-xt)^{-\beta} F\left(x\frac{1-t}{1-xt}, \frac{yt}{1-t}\right),$$

where

$$\sigma_n(y) = \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} y^k.$$

Theorem II : If there exists a generating function of the form :

$$F(x, y, t) = \sum_{n=0}^{\infty} a_n F(-n, \beta; \gamma-n; x)\, g_n(y)\, t^n$$

where $g_n(y)$ is any arbitrary polynomial, then,

(1.6) $$\sum_{n=0}^{\infty} F(-n, \beta; \gamma-n; x)\, \sigma_n(y, z)\, t^n$$

$$= (1-t)^{\gamma-1}(1-xt)^{-\beta} F\left[x\frac{1-t}{1-xt}, y, \frac{zt}{1-t}\right]$$

where,

$$\sigma_n(y, z) = \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} g_k(y)\, z^k.$$

Theorem III : If there exists a generating function of the form :

$$G(x, t) = \sum_{n=0}^{\infty} a_n F(-n, \beta ; \gamma ; x) t^n$$

then,

$$(1.7) \quad (1-y)^{\gamma-1} (1-xt)^{-\beta} G\left(x \frac{1-y}{1-xy}, yt\right) = \sum_{n=0}^{\infty} \sigma_n (x, t) y^n,$$

where

$$\sigma_n (x, t) = \sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)!} F(-n, \beta ; \gamma-n+k ; x) t^k$$

Theorem IV : If there exists a generating function of the form :

$$F(x, t) = \sum_{n=0}^{\infty} a_n F(-n, \beta-n ; \gamma-n ; x) t^n$$

then,

$$(1.8) \quad \sum_{n=0}^{\infty} F(-n, \beta-n ; \gamma-n ; x) \sigma_n (y) t^n$$

$$= (1+xt)^{\beta-\gamma} (1+t(x-1))^{\gamma-1} F\left[x+xt(x-1), \frac{yt(1+xt)}{1+(x-1)t}\right],$$

where

$$\sigma_n (y) = \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} y^k.$$

Theorem V : If there exists a generating function of the form :

$$F(x, y, t) = \sum_{n=0}^{\infty} a_n F(-n, \beta-n ; \gamma-n ; x) g_n (y) t^n$$

where $g_n (y)$ is any arbitrary polynomial of y, then,

$$(1.9) \quad \sum_{n=0}^{\infty} F(-n, \beta-n ; \gamma-n ; x) \sigma_n (y, z) t^n$$

$$= (1+xt)^{\beta-\gamma} (1+(x-1) t)^{\gamma-1} F\left[x+xt(x-1), y, \frac{zt(1+xt)}{1+(x-1) t}\right],$$

where

$$\sigma_n(y, z) = \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)\,!} g_k(y)\, z^k$$

Theorem VI : If there exists a generating function of the form :

$$G(x, t) = \sum_{n=0}^{\infty} a_n F(-n, \beta ; \gamma ; x)\, t^n$$

then

$$(1.10) \quad (1+xy)^{\beta-\gamma} (1+(x-1)\,y)^{\gamma-1} G\left(x+xy\,(x-1), \frac{yt}{1+xy}\right)$$

$$= \sum_{n=0}^{\infty} \sigma_n(x, t)\, y^n,$$

where

$$\sigma_n(x, t) = \sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} F(-n, \beta-n+k ; \gamma-n+k ; x)\, t^k$$

2. Derivation of the operators :

We know that $F(-n, \beta, \gamma, x)$ satisfies the following relations :

$$(2.1) \quad \frac{d}{dx} F(-n, \beta ; \gamma ; x) = x^{-1} (1-x)^{-1} [(\beta x-\gamma+1)\, F(-n, \beta ; \gamma ; x) + (\gamma-1)\, F(-n-1, \beta ; \gamma-1 ; x)]$$

$$(2.2) \quad \frac{d}{dx} F(-n, \beta ; \gamma ; x) = x^{-1}(1-x)^{-1} [\{x(\beta-n-1)-\gamma-1\}\, F(-n, \beta ; \gamma ; x) + (\gamma-1)\, F(-n-1, \beta-1 ; \gamma-1 ; x)]$$

Let $A = A_1 \frac{\partial}{\partial x} + A_2 \frac{\partial}{\partial y} + A_3 \frac{\partial}{\partial z} + A_4 \frac{\partial}{\partial u} + A_0$ be an operator such that

$$A[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)] = a_n\, y^{n+1} z^\beta u^{\gamma-1} F(-n-1, \beta ; \gamma-1 ; x),$$

where each A_i is a function of x, y, z, u and independent of n, β, γ and a_n is a function of n, β, γ but independent of x, y, z, u. With the help of (2.1) we have

$$(2.3) \quad A[y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)] = A_1\, x^{-1}(1-x)^{-1} y^n z^\beta u^\gamma \{(\beta x-\gamma-1)$$

$$.\, F(-n, \beta ; \gamma ; x) + (\gamma-1)\, F(-n-1, \beta ; \gamma-1 ; x)\} + y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)$$

$$.\, \{A_2\, n y^{-1} + A_3\, \beta z^{-1} + A_4\, \gamma u^{-1} + A_0\}.$$

In order to make the coefficients of $F(n-1, \beta\,;\,\gamma-1\,;\,x)\, y^{n+1} z^{\beta} u^{\gamma-1}$ independent of x, y, z, u, we choose $A_1 = x(1-x)\, yu^{-1}$, so that (2.3) reduces to

$$A[y^n z^\beta u^\gamma F(-n, \beta\,;\,\gamma\,;\,x)] = (\gamma-1) y^{n+1} z^\beta u^{\gamma-1} F(-n-1, \beta\,;\,\gamma-1\,;\,x) + [(\beta x-\gamma+1)\, yu^{-1} + A_2\, ny^{-1} + A_3\, \beta z^{-1} + A_4\, \gamma u^{-1} + A_0] \cdot y^n z^\beta u^\gamma F(-n, \beta\,;\,\gamma\,;\,x)$$

In order to make the coefficients of $y^n z^\beta u^\gamma F(-n, \beta\,;\,\gamma\,;\,x)$ zero we choose $A_2 = O$, $A_3 = -xyzu^{-1}$, $A_4 = y$, $A = -yu^{-1}$.

Thus we get

$$\text{(2.4)} \quad A = x(1-x)yu^{-1}\frac{\partial}{\partial x} - xyzu^{-1}\frac{\partial}{\partial z} + y\frac{\partial}{\partial u} - yu^{-1},$$

for which

$$\text{(2.5)} \quad A[y^n z^\beta u^\gamma F(-n, \beta\,;\,\gamma\,;\,x)] = (\gamma-1)\, y^{n+1} z^\beta u^{\gamma-1} F(-n-1, \beta\,;\,\gamma-1\,;\,x)$$

Similarly, we have on using (2.2)

$$\text{(2.6)} \quad B = x(1-x)yz^{-1}u^{-1}\frac{\partial}{\partial x} + xy^2 z^{-1} u^{-1}\frac{\partial}{\partial y} - xyu^{-1}\frac{\partial}{\partial z} + yz^{-1}\frac{\partial}{\partial u} - (1-x)\, yz^{-1}u^{-1},$$

for which

$$\text{(2.7)} \quad B[y^n z^\beta u^\gamma F(-n, \beta\,;\,\gamma\,;\,x)] = (\gamma-1)y^{n+1} z^{\beta-1} u^{\gamma-1} F(-n-1, \beta-1\,;\,\gamma-1\,;\,x)$$

3. Extended form of the groups generated by A and B :

Let $\phi_1(x, y, z, u)$ be a function such that $A\phi_1 = 0$. Then on solving $A\phi_1 = 0$ we get a solution as $\phi_1 = yz^{-1}u(1-x)$, so that A reduces to $A' = x(1-x)\, yu^{-1}\frac{\partial}{\partial x} - xyzu^{-1}\frac{\partial}{\partial z} + y\frac{\partial}{\partial u}$. Thus $A = \phi_1^{-1} A' \phi_1$

Now let X, Y, Z, U be a set of new variables for which

$$\text{(3.1)} \quad A'X = 1,\ A'Y = 0,\ A'Z = 0,\ A'U = 0,$$

so that A reduces to $\frac{\partial}{\partial X}$.

Solving (3.1) we get, a set of solutions as

$$X = \frac{u}{y},\ Y = y,\ Z = \frac{1-x}{z},\quad U = \frac{x}{(1-x)u},$$

from which we get

$$x=\frac{XYU}{1+XYU},\ y=Y,\ z=\frac{1}{Z(1+XYU)},\ u=XY.$$

Then

$$e^{aA}f(x,y,z,u)=\phi_1^{-1}(x,y,z,u)\,e^{aA'}[\phi_1(x,y,z,u)\,f(x,y,z,u)]$$

$$=\phi_1^{-1}(x,y,z,u)\exp\left(a\frac{\partial}{\partial X}\right)g_1(X,Y,Z,U)$$

$$=\phi_1^{-1}(x,y,z,u)\,g_1(X+a,Y,Z,U).$$

On calculation we get

(3.2) $$e^{aA}f(x,y,z,u)=\frac{u}{u+ay}\,f\left[x\,\frac{u+ay}{u+axy},\ y,\ \frac{zu}{u+axy},\ u+ay\right]$$

Similarly, $B\phi_2=0$ gives a solution $\phi_2=x^2yu^{-1}$, so that B reduces to B' and $B=\phi_2^{-1}B'\phi_2$. For the new set of variables X, Y, Z, U, such that

(3.3) $B'X=1,\ B'Y=0,\ B'Z=0,\ B'U=0,$

which gives a solution as,

$$x=\frac{Z+XY^2U}{Z},\ y=\frac{-Z}{XYU},\ z=-XYU,\ u=\frac{XY^2U+Z}{X^2Y^2U^2}.$$

Now

$$e^{bB}f(x,y,z,u)=\phi_2^{-1}(x,y,z,u)\,e^{bB'}[\phi_2(x,y,z,u)\,f(x,y,z,u)]$$

Thus

(3.4) $$e^{bB}f(x,y,z,u)=\frac{zu}{zu+by(1-x)}\,f\left[x\,\frac{zu+by(1-x)}{zu},\ \frac{yzu}{zu-bxy},\ \frac{zu-bxy}{u},\ u\,\frac{zu+by(1-x)}{zu-bxy}\right].$$

4. Application of the operator 'A' :

I. First we notice that

(4.1) $$e^{aA}[y^n z^\beta u^\gamma\ F(-n,\beta;\gamma;x)]=\frac{u}{u+ay}\,y^n\left(\frac{zu}{u+axy}\right)^\beta(u+ay)^\gamma\ F\left(-n,\beta;\gamma;x\,\frac{u+ay}{u+axy}\right)$$

$$=y^n z^\beta u^\gamma\left(1+\frac{ay}{u}\right)^{\gamma-1}\left(1+\frac{ay}{u}x\right)^{-\beta}F\left(-n,\beta;\gamma;x\,\frac{1+\frac{ay}{u}}{1+\frac{ay}{u}x}\right).$$

On the otherhand,

(4.2) $$e^{aA}[y^n z^\beta u^\gamma F(-n,\beta;\gamma;x)]=\sum_{m=0}^{\infty}\frac{a^m}{m!}A^m[y^n z^\beta u^\gamma F(-n,\beta;\gamma;x)]$$

$$=\sum_{m=0}^{\infty}\frac{a^m}{m!}(-1)^m(-\gamma+1)_m y^{n+m} z^\beta u^{\gamma-m} F(-n-m,\beta;\gamma-m;x).$$

Equating and using the substitution $\frac{-ay}{u}=t$, we get

(4.3) $$(1-t)^{\gamma-1}(1-xt)^{-\beta}F\left(-n,\beta;\gamma;x\frac{1-t}{1-xt}\right)$$

$$=\sum_{m=0}^{\infty}\frac{(-\gamma+1)_m}{m!}F(-n-m,\beta;\gamma-m;x)\,t^m$$

where $|t|<\min(1,|x|^{-1})$.

Now making use of the relation (4.3) we shall derive the following general theorems on generating functions :

Theorem I : If there exists a generating function of the form :

$$F(x,t)=\sum_{n=0}^{\infty}a_n F(-n,\beta;\gamma-n;x)\,t^n$$

then

(4.4) $$\sum_{n=0}^{\infty}F(-n,\beta;\gamma-n;x)\,\sigma_n(y)\,t^n$$

$$=(1-t)^{\gamma-1}(1-xt)^{-\beta}F\left(x\frac{1-t}{1-xt},\ \frac{yt}{1-t}\right),$$

where

$$\sigma_n(y)=\sum_{k=0}^{n}a_k\frac{(-\gamma+k+1)_{n-k}}{(n-k)!}y^k$$

Proof : We have

$$\sum_{n=0}^{\infty}F(-n,\beta;\gamma-n;x)\,\sigma_n(y)\,t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_k\frac{(-\gamma+k+1)_{n-k}}{(n-k)!}y^k F(-n,\beta;\gamma-n;x)\,t^n$$

$$=\sum_{k=0}^{\infty} a_k (yt)^k \sum_{n=0}^{\infty} \frac{(-\gamma+k+1)_n}{n!} F(-n-k, \beta ; \gamma-n-k ; x) t^n$$

$$=(1-t)^{\gamma-1} (1-xt)^{-\beta} \sum_{k=0}^{\infty} a_k F\left(-k, \beta ; \gamma-k ; x \frac{1-t}{1-xt}\right)\left(\frac{yt}{1-t}\right)^k$$

$$=(1-t)^{\gamma-1} (1-xt)^{-\beta} F\left[x \frac{1-t}{1-xt}, \frac{yt}{1-t}\right].$$

Theorem II : **If there exists a generating function of the form :**

$$F(x, y, t)=\sum_{n=0}^{\infty} a_n F(-n, \beta ; \gamma-n ; x) g_n(y) t^n$$

where $g_n(y)$ **is any arbitrary polynomial, then**

$$(4.5) \quad \sum_{n=0}^{\infty} F(-n, \beta ; \gamma-n ; x) \sigma_n(y, z) t^n$$

$$=(1-t)^{\gamma-1} (1-xt)^{-\beta} F\left(x \frac{1-t}{1-xt}, y, \frac{zt}{1-t}\right)$$

where $\sigma_n(y, z)=\sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} g_k(y) z^k$

Proof : **We have,**

$$\sum_{n=0}^{\infty} F(-n, \beta ; \gamma-n ; x) \sigma_n(y, z) t_n$$

$$=\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} g_k(y) z^k F(-n, \beta ; \gamma-n ; x) t^n$$

$$=\sum_{k=0}^{\infty} a_k g_k(y) (zt)^k \sum_{n=0}^{\infty} \frac{(-\gamma+k+1)_n}{n!} F(-n-k, \beta ; \gamma-n-k ; x) t^n$$

$$=(1-t)^{\gamma-1} (1-xt)^{-\beta} \sum_{k=0}^{\infty} a_k g_k(y) F\left(-k, \beta ; \gamma-k ; x \frac{1-t}{1-xt}\right)\left(\frac{zt}{1-t}\right)^k$$

$$=(1-t)^{\gamma-1} (1-xt)^{-\beta} F\left(x \frac{1-t}{1-xt}, y, \frac{zt}{1-t}\right).$$

II. Next we shall use the operator A to derive another general theorem on generating function.

Theorem III : If there exists a generating function of the form :

$$G(x, t)=\sum_{n=0}^{\infty} a_n \, F(-n, \beta ; \gamma ; x) \, t^n$$

then

$$(4.6) \quad (1-y)^{\gamma-1} (1-xy)^{-\beta} \, G\left(x \frac{1-y}{1+xy}, ty\right)=\sum_{n=0}^{\infty} \sigma_n (x, t) \, y^n$$

where

$$\sigma_n (x, t)=\sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} F(-n, \beta ; \gamma-n+k ; x) \, t^k$$

Proof : We have, $G(x, t)=\sum_{n=0}^{\infty} a_n \, F(-n, \beta ; \gamma ; x) \, t^n$.

Replacing t by ty and multiplying both sides by $z^\beta u^\gamma$, we have, on applying the operator e^{aA} to both sides,

$$e^{aA} [G(x, ty) \, z^\beta u^\gamma]=e^{aA}\left[\sum_{n=0}^{\infty} a_n \, F(-n, \beta ; \gamma ; x) \, t^n y^n z^\beta u^\gamma\right].$$

The left member becomes

$$\frac{u}{u+ay} G\left(x \frac{u+ay}{u+axy}, ty\right)\left(\frac{zu}{u+axy}\right)^\beta (u+ay)^\gamma.$$

On the other hand the right member becomes

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n t^n \frac{a^m}{m\,!} A^m [y^n z^\beta u^\gamma F(-n, \beta ; \gamma ; x)]$$

$$=\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n t^n \frac{(-a)^m}{m\,!} (-\gamma+1)_m y^{n+m} z^\beta u^{\gamma-m} F(-n-m, \beta ; \gamma-m ; x)$$

$$=\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n-m} t^{n-m} \frac{(-a)^m}{m\,!} (-\gamma+1)_m y^n z^\beta u^\gamma F(-n, \beta ; \gamma-m ; x)$$

$$=\sum_{n=0}^{\infty} y^n \sum_{m=0}^{n} a_{n-m} \frac{(-\gamma+1)_m}{m!} F(-n, \beta; \gamma-m; x)\, t^{n-m} \left(\frac{-a}{u}\right)^m z^\beta u^\gamma$$

Thus

$$\left(1+\frac{a}{u}y\right)^{\gamma-1}\left(1+\frac{a}{u}xy\right)^{-\beta} G\left(x\,\frac{1+\frac{a}{u}y}{1+\frac{a}{u}xy},\, ty\right)$$

$$=\sum_{n=0}^{\infty} \sigma_n(x, t, u)\, y^n,$$

where

$$\sigma_n(x, t, u)=\sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)!} F(-n, \beta; \gamma-n+k; x)\, t^k \left(-\frac{a}{u}\right)^{n-k}$$

Putting $-\frac{a}{u}=1$, we get

$$(1-y)^{\gamma-1}(1-xy)^{-\beta} G\left(x\,\frac{1-y}{1-xy},\, ty\right)=\sum_{n=0}^{\infty} \sigma_n(x, t)\, y^n,$$

where

$$\sigma_n(x, t)=\sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)!} F(-n, \beta; \gamma-n+k; x)\, t^k.$$

5. Application of the operator 'B':

I. First we notice that

$$(5.1)\quad e^{bB}[y^n z^\beta u^\gamma F(-n, \beta; \gamma; x)]=\frac{zu}{zu+by(1-x)}\left(\frac{yzu}{zu-bxy}\right)^n\left(\frac{zu-bxy}{u}\right)^\beta$$

$$\cdot\left[u\,\frac{zu+by(1-x)}{zu-bxy}\right]^\gamma F\left(-n, \beta; \gamma; x\,\frac{zu+by(1-x)}{zu}\right).$$

On the otherhand,

$$(5.2)\quad e^{bB}[y^n z^\beta u^\gamma F(-n, \beta; \gamma; x)]=\sum_{m=0}^{\infty}\frac{b^m}{m!} B^m y^n z^\beta u^\gamma F(-n, \beta; \gamma; x)$$

$$=\sum_{m=0}^{\infty}\frac{(-b)^m}{m!}(-\gamma+1)_m\, y^{n+m} z^{\beta-m} u^{\gamma-m} F(-n-m, \beta-m; \gamma-m; x).$$

Equating (5.1) and (5.2) and using the substitution $-\frac{hy}{zu}=t$, we get

$$(5.3)\quad (1+xt)^{\beta-\gamma-n}(1-t+xt)^{\gamma-1}\ F(-n,\beta;\gamma;x+xt(x-1))$$

$$=\sum_{m=0}^{\infty}\frac{(-\gamma+1)_m}{m!}F(-n-m,\beta-m;\gamma-m;x)\,t^m$$

where $|t| < \min(|x|^{-1}, |1-x|^{-1}, |x|^{-1}|1-x|^{-1})$.

Now making use of the relations (5.3) we shall derive two new general theorems on generating function.

Theorem IV : If there exists a generating function of the form :

$$F(x,t)=\sum_{n=0}^{\infty}a_n\,F(-n,\beta-n;\gamma-n;x)\,t^n$$

then

$$(5.4)\quad \sum_{n=0}^{\infty}F(-n,\beta-n;\gamma-n;x)\,\sigma_n(y)\,t^n$$

$$=(1+xt)^{\beta-\gamma}(1-t+xt)^{\gamma-1}\ F\left[x+xt(x-1),\frac{yt(1+xt)}{1+(x-1)t}\right]$$

where

$$\sigma_n(y)=\sum_{k=0}^{n}a_k\frac{(-\gamma+k+1)_{n-k}}{(n-k)!}\,y^k$$

Proof : We have

$$\sum_{n=0}^{\infty}F(-n,\beta-n;\gamma-n;x)\,\sigma_n(y)\,t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_k\frac{(-\gamma+k+1)_{n-k}}{(n-k)!}\,y^k\,F(-n,\beta-n;\gamma-n;x)\,t^n$$

$$=\sum_{n,k=0}^{\infty}a_k\frac{(-\gamma+k+1)_n}{n!}\,y^k\,t^{n+k}\,F(-n-k,\beta-n-k;\gamma-n-k;x)$$

$$=\sum_{k=0}^{\infty}a_k\,(yt)^k\sum_{n=0}^{\infty}\frac{(-\gamma+k+1)_n}{n!}F(-k-n,\beta-k-n;\gamma-k-n;x)\,t^n$$

$$= (1+xt)^{\beta-\gamma} (1-t+xt)^{\gamma-1} \sum_{k=0}^{\infty} a_k F(-k, \beta-k ; \gamma-k ; x+xt(x-1))$$

$$\cdot \left[\frac{yt(1+xt)}{1+(x-1)t}\right]^k$$

$$= (1+xt)^{\beta-\gamma} (1-t+xt)^{\gamma-1} F\left[x+xt(x-1), \frac{yt(1+xt)}{1+(x-1)t}\right]$$

Theorem V : If these exists a generating function of the form :

$$F(x, y, t) = \sum_{n=0}^{\infty} a_n F(-n, \beta-n ; \gamma-n ; x) g_n(y) t^n$$

where $g_n(y)$ **is any arbitary polynomial, then**

$$(5.5) \quad \sum_{n=0}^{\infty} F(-n, \beta-n ; \gamma-n ; x) \sigma_n(y, z) t^n$$

$$= (1+xt)^{\beta-\gamma} (1-t+xt)^{\gamma-1} F\left[x+xt(x-1), y, \frac{zt(1+xt)}{1+(x-1)t}\right]$$

where,

$$\sigma_n(y, z) = \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} g_k(y) z^k$$

Proof : We have

$$\sum_{n=0}^{\infty} F(-n, \beta-n ; \gamma-n ; x) \sigma_n(y, z) t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k \frac{(-\gamma+k+1)_{n-k}}{(n-k)!} g_k(y) z^k F(-n, \beta-n ; \gamma-n ; x) t^n$$

$$= \sum_{n,k=0}^{\infty} a_k \frac{(-\gamma+k+1)_n}{n!} g_k(y) z^k t^{n+k} F(-n-k, \beta-n-k ; \gamma-n-k ; x)$$

$$= \sum_{k=0}^{\infty} a_k (zt)^k g_k(y) \sum_{n=0}^{\infty} \frac{(-\gamma+k+1)_n}{n!}$$

$$\cdot F(-n-k, \beta-n-k ; \gamma-n-k ; x) t^n$$

$$=(1+xt)^{\beta-\gamma}(1-t+xt)^{\gamma-1}\sum_{k=0}^{\infty} a_k\, g_k(y)\left[\frac{zt(1+xt)}{1+(x-1)t}\right]^k$$

$$\cdot F(-k, \beta-k ; \gamma-k ; x+xt(x-1))$$

$$=(1+xt)^{\beta-\gamma}(1-t+xt)^{\gamma-1}\, F\left[x+xt(x-1), y, \frac{zt(1+xt)}{1+(x-1)t}\right]$$

II. Next we shall use the operator B to derive another new general theorem on generating function :

Theorem VI : **If there exists a generating function of the form :**

$$G(x, t)=\sum_{n=0}^{\infty} a_n\, F(-n, \beta ; \gamma ; x)\, t^n$$

then

$$(5.6)\quad (1+xy)^{\beta-\gamma}(1-y+xy)^{\gamma-1}\, G\left(x+xy(x-1), \frac{yt}{(1+xy)}\right)$$

$$=\sum_{n=0}^{\infty} \sigma_n(x, t)\, y^n$$

where,

$$\sigma_n(x, t)=\sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)!}\, F(-n, \beta-n+k ; \gamma-n+k ; x)\, t^k$$

Proof : **We have**

$$G(x, t)=\sum_{n=0}^{\infty} a_n\, F(-n, \beta ; \gamma\, x)\, t^n$$

Replacing t **by** ty **and multiplying both sides by** $z^\beta u^\gamma$, **we have, on applying operator** e^{bB} **to both sides,**

$$e^{bB}[G(x, ty)\, z^\beta u^\gamma]=e^{bB}\left[\sum_{n=0}^{\infty} a_n\, F(-n, \beta ; \gamma ; x)\, t^n y^n z^\beta u^\gamma\right]$$

The left member becomes

$$=\frac{zu}{zu+by(1-x)}\left(\frac{zu-bxy}{u}\right)^\beta\left[u\,\frac{zu+by(1-x)}{zu-bxy}\right]^\gamma$$

$$\cdot G\left[x\,\frac{zu+by(1-x)}{zu}, \frac{tyzu}{zu-bxy}\right]$$

On the other hand, the right member becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{b^m}{m\,!} \mathrm{B}^m \left[\mathrm{F}(-n, \beta\,;\,\gamma\,;\,x)\, t^n\, y^n\, z^\beta\, u^\gamma\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\, t^n \frac{(-\gamma+1)^m}{m\,!} (-b)^m\, \mathrm{F}(-n-m, \beta-m\,;\,\gamma-m\,;\,x)$$

$$. \, y^{n+m}\, z^{\beta-m}\, u^{\gamma-m}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}\, t^{n-m} \frac{(-\gamma+1)_m}{m\,!} (-b)^m\, y^n\, z^{\beta-m}\, u^{\gamma-m}$$

$$. \, \mathrm{F}(-n, \beta-m\,;\,\gamma-m\,;\,x)$$

$$= z^\beta\, u^\gamma \sum_{n=0}^{\infty} y^n \sum_{m=0}^{n} a_{n-m} \frac{(-\gamma+1)_m}{m\,!} \left(\frac{-b}{zu}\right)^m$$

$$\mathrm{F}(-n, \beta-m\,;\,\gamma-m\,;\,x)\, t^{n-m}$$

$$= z^\beta\, u^\gamma \sum_{n=0}^{\infty} \sigma_n (x, t, z, u)\, y^n,$$

where

$$\sigma_n (x, t, z, u) = \sum_{k=0}^{n} a_{n-m} \frac{(-\gamma+1)_m}{m\,!} \left(\frac{-b}{zu}\right)^m \mathrm{F}(-n, \beta-m\,;\,\gamma-m\,;\,x)\, t^{n-m}$$

$$= \sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} \mathrm{F}(-n, \beta-n+k\,;\,\gamma-n+k\,;\,x)\, t^k \left(\frac{-b}{zu}\right)^{n-k}$$

Equating and using $\frac{-b}{zu}=1$, we get,

$$(1+xy)^{\beta-\gamma}\,(1-y+xy)^{\gamma-1}\, \mathrm{G}\left[x+xy\,(x-1), \frac{ty}{1+xy}\right]$$

$$= \sum_{n=0}^{\infty} \sigma_n (x, t)\, y^n,$$

where

$$\sigma_n (x, t) = \sum_{k=0}^{n} a_k \frac{(-\gamma+1)_{n-k}}{(n-k)\,!} \mathrm{F}(-n, \beta-n+k\,;\,\gamma-n+k\,;\,x)\, t^k$$

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