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ON SOME TYPES OF AFFINE MOTIONS IN AFFINELY CONNECTED GENERALISED 2-RECURRENT SPACES

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1. Introduction : Let L_N be an affinely connected space of N-dimensions with a symmetric affine connection Γ^i_{jk} and let B^i_{jkl} $(=-B^i_{jlk})$ be the curvature tensor. Then the space is said to be a generalised 2-recurrent space if the following condition is satisfied :

$$\nabla_n \nabla_m B^i_{jkl} = a_{mn} B^i_{jkl} + \beta_n \nabla_m B^i_{jkl} \tag{1}$$

where ∇ denotes covariant differentiation with respect to Γ^i_{jk} and β_m, a_{mn} are respectively a covariant vector and a covariant tensor. Such a space shall be denoted by AG$\{^2K_N\}$ and β_m, a_{mn} will be called its vector and tensor of recurrence respectively.

We now suppose that the space admits an infinitesimal co-ordinate transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta t \tag{1'}$$

(δt being an infinitesimal constant) satisfying the condition

$$£\Gamma^i_{jk} = \nabla_k \nabla_j \xi^i + B^i_{jkl}\,\xi^l = 0 \tag{2}$$

where £ denotes Lie-derivative with respect to the above transformation. Such transformations are called affine motions.

Takano and Imai [2] considered some types of affine motions in bi-recurrent spaces. The object of this paper is to study some types of affine motions in AG$\{^2K_N\}$.

2. Some formulas in an AG$\{^2K_N\}$ admitting affine motions :

Since the space is assumed to admit affine motions the conditions (2) must be integrable. The condition of its integrability can be written as $£ B^i_{jkl} = 0$ or as

$$\xi^t \nabla_t B^i_{jkl} - B^t_{jkl} \nabla_t \xi^i + B^i_{tkl} \nabla_j \xi^t + B^i_{jtl} \nabla_k \xi^t + B^i_{jkt} \nabla_l \xi^t = 0. \tag{3}$$

Interchanging m and n in (1) and then subtracting it from (1) we get

(4) $A_{mn} B^i_{jkl} = B^t_{jkl} B^i_{tmn} - B^i_{tkl} B^t_{jmn} - B^i_{jtl} B^t_{kmn} - B^i_{jkt} B^t_{lmn}$
$$- (\beta_n \nabla_m B^i_{jkl} - \beta_m \nabla_n B^i_{jkl})$$

where $A_{mn} \equiv a_{mn} - a_{nm}$.

Putting $\nabla_j \xi^i = B^i_{jmn} f^{mn}$ where f^{mn} is a non-symmetric tensor, multiplying (4) by f^{mn} and summing over the indices m and n, we get

(5) $CB^i_{jkl} = B^t_{jkl} \nabla_t \xi^i - B^i_{tkl} \nabla_j \xi^t - B^i_{jtl} \nabla_k \xi^t - B^i_{jkt} \nabla_l \xi^t$,

where $C = A_{mn} f^{mn}$.

With the help of (5) we can express (3) as

(6) $\pounds B^i_{jkl} = \xi^t \nabla_t B^i_{jkl} - C B^i_{jkl}$

Since $\pounds B^i_{jkl} = 0$, we get

(7) $C B^i_{jkl} = \xi^t \nabla_t B^i_{jkl}$

Differentiating (7) convariantly and using (1) and (7) we have

$(\nabla_m C) B^i_{jkl} + C \nabla_m B^i_{jkl} = \nabla_m \xi^t . \nabla_t B^i_{jkl} + \xi^t (\nabla_m \nabla_t B^i_{jkl})$
$$= \nabla_m \xi^t . \nabla_t B^i_{jkl} + (\xi^t a_{tm} + C \beta_m) . B^i_{jkl}$$

or (8) $C . \nabla_m B^i_{jkl} + (\nabla_m C - C \beta_m - \xi^t a_{tm}) B^i_{jkl} = \nabla_m \xi^t \nabla_t B^i_{jkl}$

Now, multiplying (8) by ξ^m and summing with respect to m, we get

(9) $C . \xi^m \nabla_m B^i_{jkl} + (\xi^m \nabla_m C - C\, d - A) B^i_{jkl} = \nabla_m \xi^t . \nabla_t B^i_{jkl}$

where $d = \xi^t \beta_t$ and $A = \xi^l \xi^n a_{ln}$.

It is known [3] that under affine motions the operations of $\pounds$ and ∇ are interchangeable. Hence

$0 = \pounds \nabla_m B^i_{jkl} = \nabla_m \xi^t . \nabla_t B^i_{jkl} + \xi^t a_{tm} B^i_{jkl} + \beta_m \xi^t \nabla_t B^i_{jkl}$

$- \nabla_t \xi^i . \nabla^m B^t_{jkl} + \nabla_j \xi^t . \nabla_m B^i_{tkl} + \nabla_k \xi^t . \nabla_m B^i_{jtl} + \nabla_l \xi^t . \nabla_m B^i_{jkt}$

$= \nabla_m \xi^t . \nabla_t B^i_{jkl} + (\xi^t a_{tm} + C \beta_m) B^i_{jkl} - \nabla_t \xi^i . \nabla_m B^t_{jkl}$

$+ \nabla_j \xi^t . \nabla_m B^i_{tkl} + \nabla_k \xi^t . \nabla_m B^i_{jtl} + \nabla_l \xi^t . \nabla_m B^i_{jkt}$ (using (7))

Transvecting this with ξ^m and using (7) we get

(10) $\xi^m \nabla_m \xi^t . \nabla_t B^i_{jkl} + AB^i_{jkl} + C . d . B^i_{jkl}$
$$= C [\nabla_t \xi^i . B^t_{jkl} - \nabla_j \xi^t . B^i_{tkl} - \nabla_k \xi^t . B^i_{jtl} - \nabla_l \xi^t . B^i_{jkt}]$$
$$= C^2 B^i_{jkl} \quad \text{(by (3))}.$$

From (9) and (10) we have

$$\xi^m \nabla_m C = 0.$$

or

(11) $$\pounds\, C = 0.$$

3. Affine motions corresponding to a concurrent vector field :

We now consider an affine motion generated by a vector field ξ^i which is a concurrent vector field. Then

(12) $\nabla_j \xi^i = k\delta^i_j$ where k is a non-zero constant.

From (12) we have $\nabla_k \nabla_j \xi^i = 0$,

whence

(13) $$\xi^h B^i_{hjk} = -\nabla_k\nabla_j \xi^i + \nabla_j\nabla_k \xi^i = 0.$$

Differentiating (13) convariantly and using (12) we get

(14) $$k\, B^i_{mjk} + \xi^h \nabla_m B^i_{hjk} = 0$$

Again differentiating (14) covariantly and using (1) we get

$$k \nabla_n B^i_{mjk} + k \nabla_m B^i_{njk} + \xi^h (a_{mn} B^i_{hjk} + \beta_n \nabla_m B^i_{hjk}) = 0$$

or, $k (\nabla_n B^i_{mjk} + \nabla_m B^i_{njk} - \beta_n B^i_{mjk}) = 0,$ (using (13) and (14))

whence

(15) $$\nabla_n B^i_{mjk} + \nabla_m B^i_{njk} = \beta_n B^i_{mjk}$$

Now, operating ∇_l on (15) and using (1) we get

$$a_{nl} B^i_{mjk} + a_{ml} B^i_{njk} + \beta_l (\nabla_n B^i_{mjk} + \nabla_m B^i_{njk})$$
$$= \beta_n \nabla_l B^i_{mjk} + B^i_{jnjk} \cdot \nabla_l \beta_n.$$

Next, using (15) we have

$$(a_{nl} + \beta_l \beta_n - \nabla_l \beta_n) B^i_{mjk} + a_{ml} B^i_{njk} = \beta_n \nabla_l B^i_{mjk}$$

Transvecting this with ξ^m and using (13) and (14) we get

$$\xi^m a_{ml} B^i_{njk} = \xi^m \beta_n \nabla_l B^i_{mjk} = -k \beta_n B^i_{ljk}$$

Again, transvecting with ξ^n we get

$$\xi^m a_{ml} \xi^n B^i_{njk} = -k (\xi^n \beta_n) . B^i_{jlk}$$

which reduces in virtue of (13) to

$$k . \alpha . B^i_{ljk} = 0.$$

Since $k \neq 0$ and $B^i_{ijk} \neq 0$, $d = 0$ i.e. $\xi^n \beta_n = 0$.

Hence we can state the following theorem (cf. [1]):

Theorem I : If an AG{^2K_N} admits an affine motion generated by a concurrent vector field ξ^i then ξ^i, is pseudo-orthogonal to the vector of recurrence of the space.

4. Affine motions corresponding to a special concircular vector field :

Next, we consider an affine motion generated by a special concircular vector field ξ^i given by

(16) $$\nabla_j \xi^i = \phi(x)\, \delta^i_j$$

where $\phi(x)(\neq$ constant) is a scalar function of co-ordinates x^i.

At first we show that in this case the following relations hold

(i) $\phi_m \xi^m = 0$ $(\phi_m = \nabla_m \phi)$; (ii) $2\phi + C = 0$;

(iii) $A + 3\phi\, C = -\, C\,.\,d = -\, C\, \xi^l \beta_l$.

Proof of (i) :

Operating Δ_m on (16) and putting $\phi_m = \nabla_m \phi$ we get

$$\nabla_m \nabla_j \xi^i = \phi_m\, \delta^i_j$$

Also from (2), we have

$$\xi^k \nabla_k \nabla_j \xi^i = -B^i_{jkl}\, \xi^k \xi^l = 0$$

Whence, $$\xi^m \,.\, \phi^m \,.\, \delta^i_j = 0.$$

Since $\delta^i_j \neq 0$, $\xi^m \phi_m = 0$.

Pooof of (ii) :

From (3) it follows that

(1″) $$\xi^l \nabla_l B^i_{jkl} = -\, 2\phi\, B^i_{jkl}.$$

Using (7) it may be replaced by $C\, B^i_{jkl} = -\, 2\phi\, B^i_{jkl}$.

Whence $2\phi + C = 0$ [since $B^i_{jkl} \neq 0$]

Proof of (iii) :

Operating ∇_m on (1″) we obtain

$$(\xi^t a_{tm} + C\beta_m + 2\phi_m)\, B^i_{jkl} + 3\phi \,.\, \nabla_m B^i_{jkl} = (\,.$$

Multiplying this condition by ξ^m and summing on m, we get

$$(3\phi C + A + C \,.\, d)\, B^i_{jkl} = 0 \qquad \text{(using (7) and (i))}$$

or,
$$A + 3\phi C = -C \,.\, d, \qquad \text{where } A = \xi^t\, \xi^m\, a_{tm}.$$

This completes the proofs.

Now, we discuss the case of affine motion generated by a special concircular vector field ξ^i.

We have from Bianchi's second identity

$$\nabla_m B^i_{jkl} + \nabla_k B^i_{jlm} + \nabla_l B^i_{jmk} = 0$$

Covariant differentiation of this, use of (1) and this identity give

$$a_{mn}\, B^i_{jkl} + a_{kn}\, B^i_{jlm} + a_{ln}\, B^i_{jmk} = 0.$$

Multiplication with ξ^l yields

$$a_{mn}\, B^i_{jkl}\, \xi^l - a_{kn}\, B^i_{jml}\, \xi^l + a_{ln}\, \xi^l\, B^i_{jmk} = 0.$$

Applying (2) we get

(17) $$a_{ln}\, \xi^l\, B^i_{jmk} = a_{mn}\,(\nabla_k \nabla_j \xi^i) - a_{kn}\,(\nabla_m \nabla_j \xi^i)$$
$$= a_{mn}\, \phi_k\, \delta^i_j - a_{kn}\, \phi_m\, \delta^i_j = (a_{mn}\, \phi_k - a_{kn}\, \phi_m)\, \delta^i_j$$

Now, we have to consider the following two cases :

Case I : $a_{ln}\, \xi^l \neq 0$. **Case II :** $a_{ln}\, \xi^l = 0$.

Case I :

Since $B^i_{jmk} + B^i_{mkj} + B^i_{kjm} = 0$

we have
$$a_{ln}\, \xi^l\, B^i_{mjk} + a_{ln}\, \xi^l\, B^i_{mkj} + a_{ln}\, \xi^l\, B^i_{kjm} = 0$$

Using (17) this can be expressed as

$$(a_{mn}\, \phi_k - a_{kn}\, \phi_m)\, \delta^i_j + (a_{kn}\, \phi_j - a_{jn}\, \phi_k)\, \delta^i_m\, (a_{jn}\, \phi_m - a_{mn}\, \phi_j)\, \delta^i_k = 0.$$

Whence, contraction on i & j and summation over these indices yield

$$(N-2)\,(a_{mn}\, \phi_k - a_{kn}\, \phi_m) = 0.$$

Hence, for $N \geqslant 3$, $a_{mn}\, \phi_k = a_{kn}\, \phi_m$,

whence, using (i) we get

$$a_{mn}\, \xi^m\, \phi_k = 0,$$

But $a_{mn}\, \xi^m \neq 0$ by assumption. So, $\phi_k = 0$, that is, ϕ is a constant which is contrary to our assumption.

Hence, we deduce the following theorem :

Theorem 2 : There does not exist in an $AG\{^2K_N\}$ an affine motion generated by a special concircular vector field ξ^i given by $\nabla_j \xi^i = \phi(x)\,\delta^i_j$ (ϕ being a non-constant scalar) if $a_{ln}\,\xi^l \neq 0$.

Case II :

In this case, we have form (17)

$$a_{mn}\,\phi_k = a_{kn}\,\phi_m$$

So, $a_{mn}\,\xi^n\,\phi_k = a_{kn}\,\xi^n\,\phi_m$

Since $\phi_m \neq 0$, it follows that

$$a_{mn}\,\xi^n = \mu\,\phi_m$$

for a suitable scalar function μ.

However, according to (i) we have,

$$a_{mn}\,\xi^m\,\xi^n = 0$$

whence, $A = 0$.

Consequently, from (ii), (iii) we have

$$3\phi + d = 0 \qquad [\text{since } \phi \neq 0 \text{ so } c \neq 0]$$

Whence, $\phi = -\frac{1}{3}\,d = -\frac{1}{3}\,(\xi^t\,\beta_t)$.

Hence, we deduce the following theorem :

Theorem 3 : There exists in an $AG\{^2K_N\}$ an affine motion generated by a special concircular vector field ξ^i given by $\nabla_j \xi^i = \phi(x)\,\delta^i_j$ if $a_{ln}\,\xi^l = 0$ and then $\phi(x) = -\frac{1}{3}\,(c^t\,\beta_t)$.

5. Affine motions corresponding to a recurrent vector field :

We now consider an affine motion generated by a recurrent vector field ξ^i. Then $\nabla_j \xi^i = \phi_j(x)\,\xi^i$ where ϕ_i is not a gradient vector.

In this case (8) becomes

$$C\,.\,\nabla_m B^i_{jkl} + (\nabla_m C - C\,\beta_m - \xi^t\,a_{tm})\,B^i_{jkl}$$

$$= \nabla_m \xi^t\,.\,\nabla_t B^i_{jkl} = \phi_m\,\xi^t\,\nabla_t B^i_{jkl} = \phi_m\,.\,C\,.\,B^i_{jkl} \quad \text{(using (7))}$$

Hence,

$$\nabla_m B^i_{jkl} = \frac{1}{C}\,[\phi_m C - \nabla_m C + C\,\beta_m + \xi^t\,a_{tm}]\,B^i_{jkl}, \qquad C \neq 0.$$

This shows that the space is a recurrent space of first order with $\frac{1}{C}[\phi_m C - \nabla_m C + C\beta_m + \xi^t a_{tm}]$, $C \neq 0$, as its vector of recurrence.

In this case the condition (2) becomes

(18) $$\xi^i \nabla_k \phi_j + \xi^i \phi_j \phi_k = -B^i_{jkl}\, \xi^l$$

Multiplying this by ξ^k and summing over k and using $B^i_{jkl}\, \xi^k \xi^l = 0$ we get

(19) $$\xi^k \nabla_k \phi_j + \alpha \phi_j = 0 \qquad [\text{Since } \xi^i \neq 0]$$

where $\alpha = \xi^k \phi_k$.

Contracting i & k in (18) we get

$B_{jt}\, \xi^t = \alpha \phi_j + \xi^k \nabla_k \phi_j$ which reduces in virtue of (19) to

(20) $$B_{jt}\, \xi^t = 0.$$

Again contracting i and l in the Bianchi's identity

$B^i_{jkl} + B^i_{klj} + B^i_{ljk} = 0$ we get

(21) $$B_{jk} - B_{kj} + {}^{h}_{hjk} = 0.$$

Using (20) we obtain from (21)

(22) $$B_{kj}\, \xi^k = B^h_{hjk}\, \xi^k$$

Again contracting i and j in (2) we get

(23) $$\nabla_k \alpha + B^h_{hkl}\, \xi^l = 0$$

From (22) and (23) we obtain

(24) $$B_{kj}\, \xi^k = -\nabla_j \alpha$$

Contracting i and 1 in $\pounds B^i_{jkl} = 0$ we get

$$0 = \pounds B_{jk} = \xi^t \nabla_t B_{jk} + B_{tk}\, \xi^t \,.\, \phi_j + B_{jt} \xi^t \phi_k$$

or (25) $$C B_{jk} - \phi_j \,.\, \nabla_k \alpha = 0. \qquad (\text{Using } (20)\ (24) \text{ and } (71))$$

Multiplying (25) by ξ^j and summing over j and using (24) we get

$$-C \,.\, \nabla_k \alpha = \alpha \,.\, \nabla_k \alpha, \text{ that is, } (C+\alpha).\, \nabla_k \alpha = 0$$

Therefore either (i) $C + \alpha = 0$ or (ii) $\nabla_k \alpha = 0$

In case (i) $\alpha \neq 0$ because $C \neq 0$. Hence from (25) we get $-\alpha \,.\, B_{jk} = \phi_j \nabla_k \alpha$

Whence

(26) $$B_{jk} = -\phi_k \alpha_k \qquad \text{where } \alpha_k = \frac{1}{\alpha} \nabla_k \alpha.$$

In case (ii) it follows from (25) that $B_{jk}=0$.

Hence we can state the following theorem :

Theorem 4 : If an AG $\{^2K_N\}$ admits an affine motion generated by a recurrent vector field ξ^i given by $\nabla_j \xi^i = \varphi_j(x)\, \xi^i$ (ϕ_j not a gradient vector field) then the space is a recurrent space of first order and $B_{jt}\, \xi^t = 0$. Further the Ricci tensor B_{ij} is either identically zero or is of the from $B_{ij} = -\phi_i\, \alpha_j$ where $\alpha_j = \frac{1}{\alpha} \nabla_j \alpha$.

Again, differentiating (18) covariantly, we get

(27) $\xi^i \nabla_e \nabla_k \phi_j + \phi_e\, \xi^i \nabla_k \phi_j + \phi_k\, \xi^i \nabla_e \phi_j + \xi^i \phi_j \nabla_e \phi_k + \phi_j\, \phi_k\, \phi_e\, \xi^i$

$$= -B^i_{jkt}\, \phi_e\, \xi^t - \xi^t\, \nabla_e\, B^i_{jkt}$$

Also, differentiating (19) covariantly, we get

(28) $\xi^k \nabla_e \nabla_k \phi_j + \phi_e\, \xi^k \nabla_k \phi_j + \phi_j \nabla_e \alpha + \alpha \nabla_e \phi_j = 0.$

Contraction on i & k in (27) yields

$$\xi^k \nabla_l \nabla_k \phi_j + \phi_l\, \xi^k \nabla_k \phi_j + \alpha\, \Delta_l\, \phi^j + \phi_j\, \xi^t \nabla_l \phi_t + \alpha\, \phi_j\, \phi_l$$

$$= B_{jt}\, \xi^t\, \phi_l + \xi^t\, \Delta_l\, B_{jt}$$

Combining this with (28) we get

(29) $-\phi_j \nabla_l \alpha + \phi_j\, \xi^t \nabla_l \phi_t + \alpha\, \phi_j\, \phi_l = B_{jt}\, \xi^t\, \phi_l + \xi^t \nabla_l B_{jt}$

Again, covariant differentiation of $\alpha = \xi^t\, \phi_t$ gives

$$\xi^t \nabla_l \phi_t = \nabla_l \alpha - \phi_t \nabla_l \xi^t = \nabla_l \alpha - \alpha \,.\, \phi_l$$

So (29) reduces to

(30) $B_{jt}\, \xi^t \,.\, \phi_l + \xi^t \nabla_l B_{jt} = 0$

Differntiating (30) covariantly, we get

(31) $\xi^t\, \phi_l \nabla_m B_{jt} + B_{jt}\, \xi^t \nabla_m \phi_l + B_{jt}\, \phi_l\, \phi_m\, \xi^t + \phi_m\, \xi^t \nabla_l B_{jt}$

$$+ \xi^t (a_{lm} B_{jt} + \beta_m \nabla_l B_{jt}) = 0.$$

Again from (30)

(32) $B_{jt}\, \phi_l\, \phi_m\, \xi^t + \phi_m\, \xi^t\, \Delta_l\, B_{jt} = \phi_m\, [B_{jt}\, \xi^t\, \phi_l + \xi^t\, \Delta_l\, B_{jt}] = 0$

Hence (31) reduces to

$$\xi^t\, \phi_l\, \Delta_m B_{jt} + (a_{lm} + \nabla_m \phi_l)\, B_{jt}\, \xi^t + \xi^t\, \beta_m \nabla_l B_{jt} = 0$$

Using (32) we get from the above condition

(33) $(a_{lm}+\nabla_m \phi_l-\phi_l \phi_m-\beta_m \phi_l) B_{jt} \xi^t=0.$

Since $B_{jt} \xi^t=0$ (by (20)) the tensor $a_{lm}+\nabla_m \phi_l-\phi_o \phi_m-\beta_m \phi_l$ may be equal to zero and may not be so. However, we suppose that

(34) $a_{lm}+\Delta_m \phi_l-\phi_l \phi_m-\beta_m \phi_l=0,$

Hence, $\xi^l a_{lm}+\xi^l \Delta_m \phi_l-\alpha(\phi_m+\beta_m)$

$$=\xi^l (a_{lm}+\nabla_m \phi_l-\phi_l \phi_m-\beta_m\phi_l)=0.$$

Therefore, the vector of recurrence

$$K_m=\frac{1}{C}[\phi_m C-\nabla_m C+C\beta_m+\xi^t a_{tm}]$$

$$=\frac{1}{C}[\phi_m (C+2\alpha)+\beta_m (C+\alpha)-\nabla_m (C+\alpha)],\ C\neq 0.$$

By virtue of case (i) i. e. $C+\alpha=0$, K_m reduces to

(35) $K_m=-\phi_m$

Again, by virtue of case (ii) i. e. $\nabla_k \alpha=0$, K_m reduces to

(36) $K_m=\frac{1}{C}[\phi_m (C+2\alpha)+\beta_m (C+\alpha)-\nabla_m C],\ C\neq 0.$

Hence, we deduce the following theorem :—

Theorem 5 : If an A G $\{^2K_N\}$ admits an affine motion generated by a recurrent vector field ξ^i given by $\nabla_j \xi^i=\phi_j(x)\xi^i$ (ϕ_j not a gradient vector field) then the space is a recurrent space of first order and $B_{jt}\xi^t=0$. Further, the Ricci tensor B_{ij} is either identically zero or is of the form $B_{ij}=-\phi_i \alpha_j$, where $\alpha_j=\frac{1}{\alpha}\nabla_j \alpha$. In the former case, the vector of recurrence of the space is $-\phi_m$, while in the later case it is $\frac{1}{C}[\phi_m (C+2\alpha)+\beta_m (C+\alpha)-\nabla_m C]$, $C\neq 0$ provided $a_{lm}+\nabla_m \phi_l-\phi_l \phi_m-\beta_m \phi_l=0.$

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