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ON AFFINELY CONNECTED GENERALISED 2-RECURRENT SPACES

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Introduction : Generalised 2-recurrent Riemannian Spaces were studied by A. K. Roy [2] who denoted an N-space of this kind by G $\{^2K_N\}$. In the present paper we call an affinely connected space L_N with symmetric connection Γ^i_{jk} a generalised 2-recurrent space if

$$(1)\quad \nabla_p \nabla_m B^i_{jkl} = T_{mp} B^i_{jkl} + \beta_p \nabla_m B^i_{jkl}$$

where ∇ denotes covariant differentiation with respect to Γ^i_{jk} and T_{mp} is a covariant tensor of second order and β_m is a covariant vector. An N-space of this kind shall be denoted by A G $\{^2K_N\}$. It has been shown that if such a space of symmetric connection is decomposable and $T_{ij} \neq 0$, then one of the component spaces is plane. A sufficient condition is obtained in order that such a space may be an affinely connected recurrent space.

Defining an affinely connected space as a generalised projective 2-recurrent space if (1.1) $\nabla_p \nabla_m W^i_{jkl} = T^*_{mp} W^i_{jkl} + \beta^*_p \nabla_m W^i_{jkl}$ and denoting an N-space of this kind by A G P $\{^2K_N\}$ it is easy to see that every A G $\{^2K_N\}$ is an A G P $\{^2K_N\}$ but the converse is not in general true. In this paper a necessary and sufficient condition has been obtained that an A G P $\{^2K_N\}$ may be A G $\{^2K_N\}$.

1. Decomposable A G $\{^2K_N\}$

If two spaces L_M and L_{N-M} are given with co-ordinates x^α : ($\alpha, \beta, \gamma = 1, 2, \ldots, M$) and x^A : ($A, B, C = M+1, \ldots, N$) and the connections $\Gamma^\alpha_{\beta\gamma}$ and Γ^A_{BC}, then the L_N with co-ordinates x^a : ($a, b, c = 1, 2, \ldots, N$) and connection $\Gamma^a_{bc} \equiv \{\Gamma^\alpha_{\beta\gamma}, \Gamma^A_{BC}\}$ is called the product of L_M and L_{N-M}. An L_N that is a product space is said to be decomposable [1]. A geometric object in a decomposable L_N is decomposable if and only if its components with respect to the special co-ordinates are always zero when they have indices from both ranges and the components belonging to the sub-space L_N (L_{N-M}) are functions of x^α (x^A) only. In a decomposable L_N, B^a_{bcd}, B_{bc} and their covariant derivatives are decomposable.

We now consider a decomposable A G $\{^2K_N\}$.

Since (2) $\nabla_m B^i_{jkl} + \nabla_k B^i_{jlm} + \nabla_l B^i_{jmk} = 0$, we have

$$\nabla_n \nabla_m B^i_{jkl} + \nabla_n \nabla_k B^i_{jlm} + \nabla_n \nabla_l B^i_{jmk} = 0.$$

Using (1) this gives,

(3) $T_{mn} B^i_{jkl} + T_{kn} B^i_{jlm} + T_{ln} B^i_{jmk} = 0$ by virtue of (2)

Put $m=\rho, n=\sigma$; $i, j, k, l = \alpha, \beta, \gamma, \delta$

Then from (3) we get, $T_{\rho\sigma} B^\alpha_{\beta\gamma\delta} + T_{\gamma\sigma} B^\alpha_{\beta\delta\rho} + T_{\delta\sigma} B^\alpha_{\beta\rho\gamma} = 0.$

Whence (4) $T_{\rho\sigma} B^\alpha_{\beta\gamma\delta} = 0.$

Next, we put $m = \alpha, n = \beta$; $i, j, k, l = \nu, \rho, \sigma, \tau$

Then, $T_{\alpha\beta} B^\nu_{\rho\sigma\tau} + T_{\sigma\beta} B^\nu_{\rho\tau\alpha} + T_{\tau\beta} B^\nu_{\rho\alpha\sigma} = 0.$

whence (5) $T_{\alpha\beta} B^\nu_{\rho\sigma\tau} = 0.$

Now, put $m = \tau, n=\alpha'$; $i, j. k, l = \alpha, \beta, \gamma, \delta$

Then, $T_{\tau\alpha'} B^\alpha_{\beta\gamma\delta} + T_{\gamma\alpha'} B^\alpha_{\beta\delta\tau} + T_{\delta\alpha'} B^\alpha_{\beta\tau\gamma} = 0$

Whence (6) $T_{\tau\alpha'} B^\alpha_{\beta\gamma\delta} = 0.$

Finally, we put, $m = \alpha, n = \tau'$; $i, j, k, l = \nu, \rho, \sigma, \tau$

Then $T_{\alpha\tau'} B^\nu_{\rho\sigma\tau} + T_{\sigma\tau'} B^\nu_{\rho\tau\alpha} + T_{\tau\tau'} B^\nu_{\rho\alpha\sigma} = 0$

Whence (7) $T_{\alpha\tau'} B^\nu_{\rho\sigma\tau} = 0$

If $T_{ij} \neq 0$, one of $T_{\rho\sigma}, T_{\alpha\beta}, T_{\tau\alpha'}, T_{\alpha\tau'}$ must be non-null.

Hence, either $B^\alpha_{\beta\gamma\delta} = 0$ or $B^\nu_{\rho\sigma\tau} = 0.$

We can therefore, state the following theorem :

Theorem 1 : If in an A G $\{{}^2K_N\}$, $T_{ij} \neq 0$ and the space is decomposable then one of the component spaces is plane.

Henceforth, by an A G $\{{}^2K_N\}$, we shall mean a non-decomposable space.

2. Recurrent A G $\{{}^2K_N\}$: Suppose that in an A G $\{{}^2K_N\}$ the tensor T_{mp} has the form

(8) $T_{mp} = \nabla_p \chi_m + \chi_m \chi_p - \chi_m \beta_p$

where χ_m is a covariant vector field.

From (1), we get,

(9) $\nabla_p \nabla_m B^i_{jkl} = (\nabla_p \chi_m + \chi_m \chi_p - \chi_m \beta_p) B^i_{jkl} + \beta_p \nabla_m B^i_{jkl}$

If possible, let (10) $\nabla_m B^i_{jkl} = \chi_m B^i_{jkl} + H^i_{jklm}$

where $H^i_{jklm} \neq 0.$

From (10) we get,

$$\nabla_p \nabla_m B^i_{jkl} = B^i_{jkl} \nabla_p X_m + X_m \nabla_p B^i_{jkl} + \nabla_p H^i_{jklm}$$

$$= (\nabla_p X_m + X_m X_p) B^i_{jkl} + X_m H^i_{jklp} + \nabla_p H^i_{jklm}$$

Again from (9)

$$\nabla_p \nabla_m B^i_{jkl} = \beta_p X_m B^i_{jkl} + \beta_p H^i_{jklm} + (\nabla_p X_m + X_m X_p - X_m \beta_{\neq}) B^i_{jkl}$$

$$= (\nabla_p X_m + X_m X_p) B^i_{jkl} + \beta_p H^i_{jklm}.$$

Hen ce,

(11) $\nabla_p H^i_{jklm} + X_m H^i_{jklp} - \beta_p H^i_{jklm} = 0$

If the differential equations (11) have no other solution than the zero tensor, then $H^i_{jklm} \equiv 0$.

But this is contrary to the hypothesis in (10) .

Hence, from (10) it follows that the space is recurrent.

From this we get the following theorem :

Theorem 2 : If in an A G $\{^2K_N\}$ the tensor T_{mp} has the form

$$T_{mp} = \nabla_p X_m + X_m X_p - X_m \beta_p$$

and the differential equations

$$\nabla_p H^i_{jklm} + X_m H^i_{jklp} - \beta_p H^i_{jklm} = 0$$

have no other solutions than the zero tensor then the space is an affinely connected recurrent space.

3. Condition for A G P $\{^2K_N\}$ to be A G $\{^2K_N\}$

It is easy to see that every A G $\{^2K_N\}$ is an A G P $\{^2K_N\}$ with the same vector and tensor of recurrence but the converse is not in general true. Hence in this section we find a necessary and sufficient condition for the converse to be true.

Let us consider an A G P $\{^2K_N\}$ with symmetric Ricci Tensor.
Then,

$$W^i_{jkl} = B^i_{jkl} + \frac{1}{N+1} \delta^i_j (B_{kl} - B_{lk}) + \frac{1}{N^2-1}$$

$$[\delta^i_k (N B_{jl} + B_{lj}) - \delta^i_l (N B_{jk} + B_{kj})]$$

reduces to $W^i_{jkl} = B^i_{jkl} + \frac{1}{N-1}(\delta^i_k B_{jl} - \delta^i_l B_{jk})$

putting this value in (1.1) we get,

$$(12) \quad (\nabla_p \nabla_m B^i_{jkl} - \beta^*_p \nabla_m B^i_{jkl} - T_{mp}^{\ *} B^i_{jkl})$$

$$+\frac{1}{N-1}[\delta^i_k (\nabla_p \nabla_m B_{jl} - \beta^*_p \nabla_m B_{jl} - T_{mp}^{\ *} B_{jl})$$

$$- \delta^i_l (\nabla_p \nabla_m B_{jk} - \beta^*_p \nabla_m B_{jk} - T_{mp}^{\ *} B_{jk})] = 0$$

Let us now suppose that this space is affinely connected generalised 2-Ricci recurrent space.

Then, $\nabla_p \nabla_m B_{jl} - \beta^*_p \nabla_m B_{jl} - T_{mp}^{\ *} B_{jl} = 0$

Hence from (12) we have

$$\nabla_p \nabla_m B^i_{jkl} - \beta^*_p \nabla_m B^i_{jkl} - T_{mp}^{\ *} B^i_{jkl} = 0.$$

i.e. the space is affinely connected generalised 2-recurrent space.

Conversely, it is easy to see that every affinely connected generalised 2-recurrent space is affinely connected generalised 2-Ricci-recurrent space.

Hence, we deduce the Theorem :

Theorem 3 : An affinely connected generalised projective 2-recurrent space with symmetric Ricci-tensor is an affinely connected generalised 2-recurrent space if and only if it is an affinely connected generalised 2-Ricci-recurrent space.

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