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ON COMPATIBLE TOPOLOGIES OF A GROUP AND THOSE OF ITS LATTICE OF SUBGROUPS.

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Abstract :

A topology t of a group G which makes G a topological group will be called a compatible topology of G. Likewise, a topology t′ of a lattice L will be called compatible if the upper semilattice operation of L is continuous under t. It has been shown in this paper that a compatible topology of a group G induces a compatible topology of the lattice L (G) of subgroups of G. Conversely, under certain condition a compatible topology of L (G induces a compatible topology of G.

Introduction :

Topological groups as also topological lattices have been studied by many prominent Mathematicians. The aim of this paper is to study them from different point of view. As a matter of fact, an attempt has been made to connect the two studies and thereafter to study the topological groups by the lattice of compatible topologies, as introduced in this paper, of those groups.

A topology t of a group G which makes G a topological group, will be called a compatible topology of G. Likewise, a topology t′ of a lattice L will be called compatible, if the upper semilattice operation of L is continuous under t'.

Our aim is to study the topological groups by studying the corresponding lattices of the compatible topologies of the groups concerned.

In this paper we have considered the problem as to whether a compatible topology of a group G induces a compatible topology in L (G) and conversely and to this question we have a positive answer.

In fact, we have shown that a compatible topology of a group G induces a compatible topology of the lattices L (G) of subgroups of G.

Conversely, under certain condition, a compatible topology of L (G) induces a compatible topology of G.

1. **Definition :**

A set G of elements is called a generalised topological group if

(1) G is an abstract group

(2) G is a topological space.

(3) The group operations in G are continuous in the topological space G.

If the topology in G is a t_1 space i.e each point set is closed then it is called a topological group.

The set L of elements is called a topological lattice if

(1) L is a lattice

(2) L is a topological space

(3) The lattice operations are continuous in the topological space L.

We first prove the following theorem :

Theorem 1. Let G be an abstract group and L (G) be the lattice of all subgroups of G. If G be a generalised topological group, then the topology of G induces a topology in L (G) for which L (G) is a topological upper semilattice.

Proof : Let G be a generalised topological group. Let Σ be the complete system of neighbourhoods of the topological space G.

Let $U \epsilon \Sigma$. We shall denote by U* the smallest sugroup of G generated by U.

Correspondingly, for $V \epsilon \Sigma$ one gets V*. Let $W^* = U^* \bar{\cup} V^*$ = smallest subgroup of G containing U* and V*.

We say that W* is the union of U* and V* and we shall denote this union by $\bar{\cup}$, to distinguish it from the set union.

We consider the set $\Sigma^* = \{\bar{\cup}\, U^*,\ \forall U \epsilon \Sigma\}$
we shall show that this Σ^* becomes the complete system of neighbourhoods of a topology in L (G) for which L (G) becomes a topological upper semilattice.

Let $H \epsilon L(G)$.

Then H must have a system of generators, say, s.

Hence there exists an open set U | $S \subset U$ and from U we get U^*. So, $H \in U^*$

Next, Σ^* satislies t e following conditions :—

1) $L(G) = \bigcup U^*, \forall U^* \in \Sigma^*$.

It is obvious, as for every $H \in L(G)$, $\exists U^* \in \Sigma^*$ | $H \in U^*$ and so, $L(G) = \bigcup U^*, \forall U^* \in \Sigma^*$.

(2) For any two sets U^* and $V^* \in \Sigma^*$ which contain the subgroup $H \in L(G)$, there exists a $W^* \in \Sigma^*$ | $H \in W^* \subseteq^* U^* \cap^* V^*$.

It is obvious that U^* and V_* are open sets of G and as $H \in U^* \cap V^* \Rightarrow \exists$ open set W containing a system of generators S of H | $S \subseteq W$ and $W \subseteq V^* \cap V^* \Rightarrow H \in W^* \subseteq U^* \cap V^*$.

HenceΣ^* is a complete system of neighbourhood of the topological space L (G).

The upper lattice oparation is continuous i.e.

(A) If $H_1 \bar{\cup} H_2 \in U_* \Rightarrow \exists H_1 \in U_1^*$ and $H_2 \in U_2^*$ | $U_1^* \bar{\cup} U_2^* \subseteq U^*$ where H_1 and H_2 are any two subgroups of G.

Let $H_1 \bar{\cup} H_2 \in U^*$

Let S_1 and S_2 are two system of generators of H_1 and H_2 respectively.

Hence we can find two open sets $S_1 \subseteq U_1$ and $S_2 \subseteq U_2$ $U_1 \subset U^*$ and $U_2 \subset U^*$.

From U_1 and U_2 we get U_1^* and U_2^* and $H_1 \in U_1*$, $H_2 \in U_2^*$ | $U_1^* \bar{\cup} U_2^* \subseteq U^*$.

Hence L (G) is a topological upper semilattice.

We note that Σ^* satisfies the following condition :

(B) Let $a \in G$ and $U^* \in \Sigma^*$, then we can find a $V^* \in Z^*$ | $aV^*a^{-1} \subset U^*$.

As, U^* is an open set in G and identity $e \in U^*$ it follows that there exists a neighbourhood V of e such that for any element 'a' $\in G$, $aVa^{-1} \subset U^* \Rightarrow aV^*a^{-1} \subset U^*$.

2. **Theorem 2 :** If L (G) be a topological upper semilattice, the complete system of neighbourhoods of which satisfies the condition (B), then the topology of L (G) induces a topology in G, for which G becomes a generalised topological group.

Proof ; Let Σ^* be the complete system of neighbourhood of L (G) satisfying condition (B).

Let $U^* \in \Sigma^*$

Let U_i be the set of all subgroups of G contained in U^*.

Let U' be the set of all elements of $\bigcup u_i$

Let $\Sigma' = \{U', \forall U^* \in \Sigma^*\}$.

It can be easily shown that Σ' satisfies the following conditions :

(1) Identity is a common element to all the sets.

(2) The intersection of any two sets belonging to Σ' contains a third set of the system Σ'.

(3) For any set U' of the system Σ' there exists a set $V' \epsilon \Sigma'$ such that $V'V'^{-1} \subset U'$

(4) For any set U' of the system Σ' and an element $a \epsilon U'$ there exists a set $V' \epsilon \Sigma' \mid V'a \subset U'$

(5) If $U' \epsilon \Sigma'$ and $a \epsilon G$, there exists a set $V' \epsilon \Sigma'$ such that $a^{-1}V'a \subset U'$.

(1), (2), (4) and (5) are obvious.

For (3), let $U' \epsilon \Sigma'$. We get a $U^* \epsilon \Sigma^*$, Let $H \epsilon U^*$

Now, $H.H = H \bar{\cup} H = H$.

As this lattice operation is continuous, for every neighbourhood U^* of H, there exists a neighbourhood V^* of H such that $V^*.V^* = V^* \bar{\cup} V^* \subseteq U^*$. Thus $V'V'^{-1} \subseteq U'$

Thus Σ' satisfies all the conditions stated above.

Hence Σ' is a complete system of neighbourhood of the identity and G is a generalised topological group.

3. A topology of the group G for which the group operations are continuous will be called compatible, similarly, the topologies of L (G) for which the upper semilattice operation is continuous will be called compatible in the lattice.

Thus, from a compatible topology t in G we get a compatible topology t^*, say, in L (G) and from the compatible topology t^* in L (G), we get a compatible topology, say, t' in G.

Preposition 1 : $t' \leqslant t$.

It is obvious.

Proposition 2 : (a) $t_1 \leqslant t_2 \Rightarrow t_1^* \leqslant t_2^*$

(b) $t_1^* \leqslant t_2^* \Rightarrow t_1' \leqslant t_2'$.

Proof : (a) Let $t_1 \leqslant t_2$.

Let Σ_1 and Σ_2 are the complete system of neighbourhoods of the topologies t_1 and t_2 respectively and let Σ_1^* and Σ_2^* be the complete system of neighbourhoods of t_1^* and t_2^* obtained from Σ_1 and Σ_2 respectively.

Let $a \epsilon G$ and let $H = (a)^*$ be the cyclic subgroup generated by a.

Then $(a)^* \epsilon L$ (G). Let $a \epsilon U_1 \epsilon t_1$. From U_1 we get U_1^* and so, $H \epsilon U_1^*$.

As, $t_1 \leqslant t_2$, therefore, $\exists$ $U_2 \mid a \epsilon U_2 \subseteq U_1 \Rightarrow$
$H = (a)^* \epsilon U_2^* \subseteq U_1^* \Rightarrow t_1^* \leqslant t_2^*$.

(b) It can be easily proved.

Theorem 3 : The set T of all compatible topologies of G, is a complete laftice.

Proof : Now, the weakest topology $J=\{G, \phi\}$ where ϕ is the empty set, is a compatible topology.

Let $t_1, t_2 \epsilon T$.

But $t_1 \cap t_2$ we shall mean the largest compatible topology of G contained in t_1 and t_2.

Let $\tau \subseteq T$ be a non-empty set of topologies in T.

Then $\bigcap_{ti \in \tau} ti \in T$, as $J \in T$.

As the discete topology is the laagest compatible topology in G, it follows that T is a complete lettice.

Theorem 4 : The set L (T) of all compatible topologies of L (G) is a complete lattice.

Proof : Now, the weakest topology $\bar{J} = \{L(G), \phi\} \in L(T)$.

Also the discrete topology belongs to L (T) Thus if t_1 and $t_2 \in L(T)$, then by $t_1 \cap t_2$ we shall mean the largest compatible topology contained in t_1 and t_2. Then as in the above case, L (T) is a complete lattice.

4. The construction of the eystem Σ^* of neighbourhoods of the compatible topoloy t^* of L (G), from a compatible topology t of G, given by the system of neighboutrhoobs Σ, as in theorem 1, is unique.

Hence one can define a mapping $f : T \to L(T)$ as $f(t) = t^*, \forall t \in T$. This mapping f is isotone, as has been seen in Prop 2.

Also, the compatible topology t' of G, obtained from a compatible topology t^* of L (G), as in Theorem 2, is unique. Hence we cao define a mapping $f' : L(T) \to T$ by $f'(t^*) = t', \forall t^* \in L(T)$.

This f' is isotone, by Prop. 2.

We define a relation ρ in T such that $t_1 \rho t_2$, iff $f(t_1) = f(t_2)$ holds in L (T).

ρ is an equivalence relation.

We shall show that ρ is a congruence for the lower semilattice operation.

Let $t_1 \rho t_2$ and $\bar{t}_1 \rho \bar{t}_2$ holds.

Then $f(t_1) = f(t_2)$ and $f(\bar{t}_1) = f(\bar{t}_2)$.

As, $t_1 \cap \bar{t}_1 \leqslant t_1, \bar{t}_1$, $f(t_1 \cap \bar{t}_1) \leqslant f(t_1), f(\bar{t}_1)$ by Prop. 2.

i. e. $f(\bar{t}_1 \cap t_1) \leqslant f(t_1) \cap f(\bar{t}_1)$ by Theorem 4. ... (1)

So, $f'(f(t_1 \cap \bar{t}_1)) \leqslant f'(f(t_1) \cap f(\bar{t}_1))$

Hence, $f'(f(t_1) \cap f(\bar{t}_1)) \leqslant f'(f(t_1)), f'(f(\bar{t}_1)) \leqslant t_1, \bar{t}_1$

So, $f'(f(t_1) \cap f(\bar{t}_1)) \leqslant t_1 \cap \bar{t}_1$

Hence, $f(t_1) \cap f(\bar{t}_1) \leqslant f(t_1 \cap \bar{t}_1)$ by Prop. 2. ... (2)

From (1) and (2) it follows that

$f(t_1 \cap \bar{t}_1) = f(t_1) \cap f(t_1)$ and similarly, it can be shown that $f(t_2 \cap \bar{t}_2) = f(t_2) \cap f(\bar{t}_2)$.

Hence it follows that $f\, t_2 \cap \bar{t}_2) = f(t_2) \cap f(\bar{t}_2 = f(t_1 \cap \bar{t}_1)$

Hence, ρ is a congruence relation for the lower-semilattice. Thus we have :

Theorem 5 : The lower semilattice T/ρ is monomorphic to L (T).

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