Jour. Pure Math.
Vol. 5 (1985—86), pp. 73—80

SOME RESULTS ON THE DISTANCE SET OF THE CANTOR TYPE SET C_k

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ABSTRACT

In this paper we construct a class of linear sets C_k, which includes the classical Cantor set $C(=C_1)$ and whose construction and properties are very much similar to those of C. Also we have studied some properties of the distance set as well as mid-point set of C_k.

INTRODUCTION :

In 1917, Hugo Steinhaus [9] proved the remarkable property that the distance set of the Cantor set in the unit interval is precisely the interval [0, 1] ; and in 1920, he [10] demonstrated that the distance set of any set with positive Lebesgue measure contains an interval with left end point zero. This result has also been established using alternative methods by S. Ruziewicz [7], T. Šalăt and T. Neubrunn [8], Bose Majumder [1] J. Randolph [6]. In 1947, H. Kestelman [4] considered p-dimensional sets A with the property that every sufficiently small vector in Euclidean p-space is the difference of two elements of A i. e. the "directed distances" of A contains a sphere and hence Steinhaus' result is a particular case of Kestelman's result for $p=1$. Utz [11] has also proved the result $D(C)=[0, 1]$ by geometrical way.

In this paper we construct a class of linear sets $\{C_k\}$, which includes the classical Cantor set $C(=C_1)$ and whose construction and properties are very much similar to those of C We have also studied some properties of the distance set of C_k.

DEFINITIONS AND NOTATIONS :

(1) The distance set of $A(\subset R)$ denoted by $D(A)$ is the set

$$D(A)=\{\,|x-y|\ /x\in A,\ y\in A\}.$$

(2) The mid point set of $A(\subset R)$ denoted by $M(A)$ is the set

$$M(A)=\left\{\frac{x+y}{2} \,/ x\epsilon A,\ y\epsilon A\right\}$$

§1. **CONSTRUCTION OF A LINEAR SET** (for a given positive integer k) in the closed interval [0, 1].

We first divide the interval [0, 1] into (2k+1) equal parts and remove the open intervals in the second, fourth, sixth, ..., 2k th positions leaving (k+1) closed intervals each of length $\frac{1}{2k+1}$. We shall call each of these remaining closed intervals class 1 intervals and denote each by A_1. Let $E_1=UA_1$. Each A_1 is again divided into (2k+1) equal parts and the open intervals in the second, fourth, sixth, ..., 2k th positions are again deleted, leaving $(k+1)^2$ closed intervals each of length $\frac{1}{(2k+1)^2}$. We call each of these remaining closed intervals class 2 closed intervals and each denoted by A_2. Let $E_2=UA_2$. This process is continued inductively. During the n^{th} stage we delete the open intervals at the second, fourth, ..., 2k th position each of length $\frac{1}{(2k+1)^n}$ and denote each by A_n. Let $E_n=UA_n$. Then $E_n(n=1, 2, 3, \ldots)$ form a monotone decreasing sequence of compact sets and thus, have nonempty intersection. Let $C_k=\bigcap_{n=1}^{\infty} E_n$. The set C_k, being the complement of an everywhere dense set of non-overlapping, non-abutting, open intervals is a non-dense perfect set [3]. Now, total length suppressed at the first stage$=\frac{k}{2k+1}$.

Number of closed intervals left at the first stage=(k+1).

Total length suppressed at the second stage$=\frac{k}{(2k+1)^2}(k+1)$

Thus, total length suppressed at the nth stage$=\frac{k(k+1)^{n-1}}{(2k+1)^n}$

Hence total length removed

$$=\frac{k}{(2k+1)}+\frac{k(k+1)}{(2k+1)^2}+\frac{k(k+1)^2}{(2k+1)^3}+\ldots\ldots\ldots+\frac{k(k+1)^{n-1}}{(2k+1)^n}+\ldots\ldots$$

Hence the Lebesque measure of $C_k=0$.

The set C_k for a given positive integer k may also be described in series notations as the set of all x such that

$$x=\sum_{i=1}^{\infty} \frac{a_i}{(2k+1)^i}, \quad a_i=(0, 2, 4, \ldots\ldots, 2k) \text{ for all } i.$$

It is easy to see that the set C_k is symmetric i. e. if $x \epsilon C_k$ then $1-x \epsilon C_k$.

D. Ganguly [2] proved that $D(C_k)=[0, 1]$. We shall present an alternative proof of this result which seems to be shorter.

THEOREM : 1. 1. $D(C_k)=[0, 1]$.

PROOF : Let $K=\{x-y/x \epsilon C_k, y \epsilon C_k\}$.

To prove this result it is sufficient to show that $K=[-1, 1]$.

Let $x=\sum_{i=1}^{\infty} \frac{a_i}{(2k+1)^i}$ and $y=\sum_{i=1}^{\infty} \frac{b_i}{(2k+1)^i}$

where $a_i, b_i=\{0, 2, 4, \ldots\ldots, 2k\}$ for each i.

Since $1=\sum_{i=1}^{\infty} \frac{2k}{(2k+1)^i}$, hence

$$x-y+1=\sum_{i=1}^{\infty} \frac{a_i-b_i+2k}{(2k+1)^i}=\sum_{i=1}^{\infty} \frac{2c_i}{(2k+1)^i}$$

where $2c_i=a_i-b_i+2k$, $c_i=\{0, 1, 2, \ldots \ldots, 2k\}$.

Thus $\frac{x-y+1}{2}=\sum_{i=1}^{\infty} \frac{c_i}{(2k+1)^i}$ is any point in $[0, 1]$.

Thus every $p \epsilon [0, 2]$ may be expressed as $p=x-y+1$, where $x, y \epsilon C_k$.

Thus $[0, 2]=K+1$ and hence $K=[-1, 1]$.

Theorem 1. 2: If d is any point in $[0, 1]$ then there exists a pair of points from C_k whose mid point is d i.e. $M(C_k)=[0, 1]$.

Proof : Let $0 \leqslant d \leqslant 1$. Then $1-d \epsilon [0, 1]$. By theorem 1. 1 there exist x and y in C_k such that $y-x=1-d$.

Hence $(1-y)+x=d$.

As C_k is symmetric hence $1-y \epsilon C_k$. Thus given any $d \epsilon [0, 1]$ there exist x and y in C_k whose sum is d.

If $0 \leq 2d \leqslant 1$, then there exist x and y in C_k such that $x+y=2d$ i.e. $d=\frac{x+y}{2}$.

If $1 \leqslant 2d \leqslant 2$, then $0 \leq 2-2d \leqslant 1$ and hence there exists a pair of points $x, y \epsilon C_k$ such that $x+y=2-2d$.

Then $(1-x)+(1-y)=2d$

or $x'+y'=2d$ where $x'=1-x\epsilon C_k$

and $y'=1-y\epsilon C_k$.

Therefore, in any case, for every $d\epsilon[0,1]$ there is a pair of points from C_k whose middle point is d.

We shall now generalize the result 1. 2. Before going to prove the generalized result we shall now state a result due to Ganguly [2].

Result. Given any real number d and m such that $0 \leqslant d \leqslant 1$ and $\frac{1}{2k+1} \leq |m| \leq 2k+1$, there exists at least one pair of points $(x, y)\epsilon C_k x C_k$ s.t $y=mx+d$.

Theorem 1. 3. Given two positive real numbers μ and ν satisfying $\frac{1}{2k+1} \leq \frac{\mu}{\nu} \leq 1$, each point d in $0 \leq d \leq 1$, divides a segment $[x, y] \subseteq [0, 1]$ in the ratio $\nu : \mu$ whose end points x and y are the points of C_k.

Proof : Let d be any point in $0 < d \leq \frac{\nu}{\mu+\nu}$; we now choose d' such that

$$d' = \left[\frac{\mu+\nu}{\nu}\right] d.$$

Hence $d=\frac{\nu d'}{\mu+\nu}$.

Since $0 < d \leqslant \frac{\nu}{\mu+\nu}$, we have $0 < d' \leqslant 1$. We now choose $m=-\frac{\mu}{\nu}$ in above type of result on C_k.

Therefore $\frac{1}{2k+1} \leq |m| \leq 1 \; (< 2k+1)$ and thus $y=(-\mu/\nu)x+d'$, where $x\epsilon C_k$ and $y\epsilon C_k$.

Hence $\frac{\nu y+\mu x}{\nu}=d'$

or, $\frac{\nu y+\mu x}{\nu}=\frac{\mu+\nu}{\nu}d$

or, $d=\frac{\nu y+\mu x}{\mu+\nu}$.

Taking $\frac{\nu}{\mu+\nu} < d < 1$, we get

$$1-\frac{\nu}{\mu-\nu} > 1-d > 0, \; 0 < 1-d < \frac{\mu}{\mu+\nu}\left(< \frac{\nu}{\mu+\nu}\right).$$

Hence, by previous argument, we get

$x \epsilon C_k$, $y \epsilon C_k$ and $1-d=\dfrac{\nu y+\mu x}{\mu+\nu}$

or, $\mu+\nu-d(\mu+\nu)=\nu y+\mu x$

or, $(\mu+\nu)d=\mu(1-x)+\nu(1-y)=\mu x'+\nu y'$

where $x'=(1-x)\epsilon C_k$

and $y'=(1-y)\epsilon C_k$.

Thus $d=\dfrac{\mu x'+\nu y'}{\mu+\nu}$

where $\dfrac{\nu}{\mu+\nu}<d<1.$

Hence the result follows.

§ 2. We are now interested to determine for a given $d \epsilon [0, 1]$ how many pairs of points x and y belonging to C_k are there such that $d=y-x$.

Let $T=C_k X C_k$ and let l denote the line $y=x+d$, $0 \leqslant d \leqslant 1$. Since $D(C_k)=[0, 1]$ the line $y=x+d$ must intersect T at least in one point.

Definition : Given $0 \leqq d \leq 1$, we define $\triangle_d$ to be the set

$\{(x, y) \mid x \epsilon C_k,\ y \epsilon C_k,\ y-x=d\}$

Note that whenever $y-x=d$, then $|y-x|=d$ but also $|x-y|=d$, and the pair $(y, x) \bar{\epsilon}\ \triangle_d$. $\overset{=}{\triangle}_d$ describes precisely the number of distinct pairs of Cantor type points with the property that they are d unit apart.

Theorem 2.1 : For all but a countable number of d in C_k, $\overset{=}{\triangle}_d=c$.

Proof : Let $d \epsilon C_k$ such that d is not an end point of deleted intervals in the construction of C_k. Then we can express d as

$$d=\sum_{i=1}^{\infty} \frac{a_i}{(2k+1)^i}$$

where $a_i=(0, 2, 4, \ldots, 2k)$ for all i, and each of the values $0, 2, 4, \ldots, 2k$ occurs infinitely many times.

Let x be a number expressed in the scale of $(2k+1)$ such that

$$x=\sum_{i=1}^{\infty} \frac{x_i}{(2k+1)^i},$$

where

$x_i=0$ if $a_i=(2, 4, \ldots, 2k)$

and

$x_i=(2, 4, \ldots, 2k)$ if $a_i=0$.

Then the number of distinct x's for a given d, attainable in this manner is the number of sequences 0's, 2's, ... 2k's i. e. c (cardinal number of the continum). According to the construction of x, we can say $x \epsilon C_k$. If we let $y=x+d$, then since x and d never have the digit 2, 4, ..., 2k in the same position, $y \epsilon C_k$. Hence $\bar{\bar{\triangle}}_d=c$.

Theorem 2.2 : For a dense set of $d \bar{\epsilon} C_k$, $\bar{\bar{\triangle}}_d=c$.

Proof : It can be easily proved that the numbers having a terminating expansion of the scale $(2k+1)$ form a dense set in the complement of C_k in [0, 1]. Let d be one such number. There exists at least one pair of points (x, y) in Cantor type set C_k such that $y-x=d$. As d terminates, x and y must have identical digits from some index m onwards when they are expressed in the scale of $(2k+1)$.

$$\text{Then } x = \sum_{i=1}^{\infty} \frac{a_i}{(2k+1)^i} \quad \text{and } y = \sum_{i=1}^{\infty} \frac{b_i}{(2k+1)^i}$$

where $a_i, b_i=(0, 2, 4, \ldots, 2k)$ with the condition $a_i=b_i$ for $i>m$.

We can choose a_i and b_i in $(k+1)$ ways so that $a_i=b_i$. Therefore the number of pairs of points (x, y) satisfying $x \epsilon C_k$ and $y \epsilon C_k$ and $y-x=d$ is $(k+1)^{\aleph_0}=c$ [5]. Hence $\bar{\bar{\triangle}}_d=c$ for a dense set of $d \bar{\epsilon} C_k$.

We shall now state the following theorem which is proved by Ganguly to determine the number of pairs of points of C_k associated with almost every $d \epsilon [-1, 1]$ such that $d=y-x$.

Theorem 2.3 : For almost all $d \epsilon [0, 1]$, $\bar{\bar{\triangle}}_d=c$.

We now elaborate the theorem by

Theorem 2.4 : For every $d \epsilon [0, 1]$, the set $\triangle_d$ is either finite or perfect set.

The following lemma is needed to establish the theorem.

LEMMA : For every d in [0, 1], $\triangle_d$ is a closed subset of the unit square.

Proof : Let $A=\{(x, y)/x \epsilon C_k, y \epsilon C_k \text{ and } y>x\}$. Let $f : A \to [0, 1]$ be a function defined by $f(x, y)=y-x$. Then obviously f is continuous. Let d be any point in the unit interval [0, 1]. Then $f^{-1}\{d\}=\triangle_d$ and hence $\triangle_d$ is a closed set.

Proof of the theorem : By the lemma $\triangle_d$ is closed. We have to prove that $\triangle_d$ is dense-in-itself. It has been proved by Gangaly [2] if there are an infinite number of pairs of points $x \in C_k$, $y \in C_k$ such that $y-x=d$ for a given $d \in [0, 1]$ then each pair, when expressed In the form

$$x=\sum_{i=1}^{\infty} \frac{2\alpha_i}{(2k+1)^i},\ y=\sum_{i=1}^{\infty} \frac{2\beta_i}{(2k+1)^i},\ \alpha_i, \beta_i = \begin{Bmatrix} 0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ k \end{Bmatrix} \quad \ldots(1)$$

has the property that $\alpha_i=\beta_i$ for infinitely many i.

Suppose $\triangle_d$ is an infinite set for a given $d \in [0, 1]$. Let (x, y) be any element of $\triangle_d$, where x and y are expressed in the form of (1).

Let $\epsilon^*(>0)$ be chosen prevlously. Then we choose a positive integer N such that $\frac{2}{(2k+1)^N} < \sqrt{\frac{\epsilon}{2}}$ and $\alpha_N = \beta_N$. If $\alpha_N = \beta_N = 0$, then $x + \frac{2}{(2k+1)^N}$ and $y + \frac{2}{(2k+1)^N}$ are the points of C_k. If $\alpha_N = \beta_N = (1,2,3 \ldots, k)$.

then

$$x - \frac{2}{(2k+1)^N} \text{ and } y - \frac{2}{(2k+1)^N} \text{ are the points of } C_k.$$

Hence $\left| \left(y \pm \frac{2}{(2k+1)^N}\right) - \left(x \pm \frac{2}{(2k+1)^N}\right) \right| = |y-x| = d$

Therefore $\left(x + \frac{2}{(2k+1)^N},\ y + \frac{2}{(2k+1)^N}\right)$

or $\left(x - \frac{2}{(2k+1)^N},\ y - \frac{2}{(2k+1)^N}\right)$ is an element of $\triangle_d$. Also

$$\left[\left(y \pm \frac{2}{(2k+1)^N}\right) - y\right]^2 + \left[\left(x \pm \frac{2}{(2k+1)^N}\right) - x\right]^2$$

$$= 2 \left(\frac{2}{(2k+1)^N}\right)^2 < \epsilon.$$

Thus (x, y) is a limit point of $\triangle_d$.

Hence the theorem.

REFERENCES

[1] Bose Majumder, N. C.: 'On the distance set of the Cantor middle third set' Bull. Cal. Math. Soc. 51 (1959) pp 93—102.

[2] Ganguly D. K.: Study of some properties of functions and sets connected with cantorset and similer type Ph.D. Thesis, 1979, Calcutta University.

[3] Hobson, E. W.: "Theory of Function of a real variable" Vol. I Dover Publication Inc. N.Y. p 117

[4] Kestelman, H.: "The convergent sequence belonging to a set' J. Lond. Math. Soc. 22 (1947), pp 130—136.

[5] Kamke, E.: "Theory of sets' Dover Publication Inc. N.Y. p 49.

[6] Randolph, J. F.: "Distance between points of the Cantor set" Amer. Math. Monthly, 47 (1940), pp. 549—551.

[7] Ruziewicz, S.: "Contribution a Le'tude des ensembles de distances de points' Fund. Math 7 (1925), pp. 141—143.

[8] Šalát, T. and Neubrunn, T.: "Distance sets, Ratio sets and certain transformations of sets of Real number" Cas. pro. pes. mat. 94 (1969) pp. 381—393.

[9] Steinhaus, H,: "Nowa Vlanosĕ Monogoscĭ G. Cantor Wecktor, 1917, pp. 105—107.

[10] Steinhaus, H.: "Sur les distances des points dans les ensembles de measure positive' Fund. Math. (1920), pp. 93—104.

[11] Utz, W.R.: "The distance set for the Cantor discontinuum' Amer. Math. Monthly 58 (1951). pp. 407—408.

Received
10. 7. 1986

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