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ON NORMAL PSEUDO-IDEALS IN SEMIGROUPS.

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1. Introduction.

A semigroup S is called normal if Sx=xS for all elements x of S [7]. A pseudo-ideal A of a semigroup S is called normal if A is a normal subsemigroup of S i. e. if xA=Ax for all $x \in A$. In this paper we have studied some properties of normal pseudo-ideals. The purpose of this paper is to give some properties which characterise normal semigroups and normal regular semigroups in terms of normal pseudo-ideals and bi-ideals.

Let NB (S) denote the set of all normal pseudo-ideals and bi-ideals of a semigroup S. Then NB (S) is a semigroup under the multiplication of subsets and N (S), the set of all normal pseudo-ideals of S is a commutative subsemigroup of NB (S). We have shown that a semig oup S is normal if and only if NB (S) is normal. Lastly we have characterised regula ity of all those semigroups in which all the pseudo-ideals are normal.

2. In [8] Sen has shown that in a group every pseudo-ideal is a normal pseudo-ideal ; the result is also true in a commutative semigroup. The following example shows that there are also semigroups which are neither groups nor commutative semigroups but contain normal pseudo-ideals.

2. 1 Example. Let S=G X J where G is a noncommutative group and J is the set of all integers ; then S is a semigroup with respect to the multiplication defined componentwise. Let $A=G \times J^+$ be a subset of S where J^+ denotes the set of all nonnegative integers Then A is a pseudo-ideal of S. Also for any element x of S, xA=Ax. So A is a normal pseudo ideal of S.

2 2 Proposition. A normal subsemigroup A of a semigroup S will be a pseudo-ideal if and only if $xAx \subseteq A$ for every $x \in S$.

Proof. Let a normal subsemigroup A of a semigroup S be a pseudo-ideal of S Then $xAx = xxA = x^2A \subseteq A$. On the other hand if for a normal subsemigroup A of a semigroup S, $xAx \subseteq A$ then $x^2A = xxA = xAx \subseteq A$ for every $x \in S$. Similarly $Ax^2 \subseteq A$ for every $x \in S$. So A is a pseudo-ideal of S.

2. 3 Proposition. Every one-sided normal pseudo-ideal of a semigroup S is a pseudo-ideal (two-sided) of S.

Proof. Let A be a normal left pseudo-ideal of a semigroup S and $x \in S$. Then $Ax^2 = x^2A \subseteq A$. So A is also a right pseudo-ideal of S. Similarly we can show that if A is a right pseudo-ideal then A is also a left pseudo-ideal. Hence the proposition.

2. 4 Proposition. Let N (S) denote the set of all normal pseudo-ideals of a semigroup S ; then N (S) is a commutative semigroup under the multiplication of subsets.

Proof. Let A_1, $A_2 \in N(S)$. Let a_1a_2, $b_1b_2 \in A_1A_2$ where a_1, $b_1 \in A_1$ and $a_2, b_2 \in A_2$. Then $a_1a_2b_1b_2 = a_1b_1a_2'b_2 \in A_1A_2$ where $a_2b_1 = b_1a_2'$, $a_2' \in A_2$ and also for any $x \in S$ $x^2 (A_1A_2) = (x^2A_1) A_2 \subseteq A_1 A_2$. So $A_1 A_2$ is a left pseudo-ideal of S. Since A_1, A_2 are both normal, $x(A_1A_2) = (A_1 A_2)x$. Hence $A_1 A_2 \in N(S)$. Lastly let $a_1 \in A_1$ then $a_1 A_2 = A_2 a_1$ So $A_1 A_2 = A_2 A_1$. Evidently $A_1 (A_2 A_3) = (A_1 A_2) A_3$. Consequently N (S) is a commutative semigroup.

2. 5 Proposition. Let NB (S) denote the set of all normal pseudo-ideals and bi-ideals of a semigroup S ; NB (S) is a semigroup under the multiplication of subsets.

Proof. It follows from the above proposition that the product of two normal pseudo-ideals is a normal pseudo-ideal. Also the product of two bi-ideals of S is a bi-ideal [3]. Let $A \in N(S)$ and $B \in B(S)$, the set of all bi-ideals of S. Now $(AB)(AB) = A(BAB) \subseteq AB$ and $(AB) S (AB) = AB (SA) B \subseteq AB$. So $AB \in B(S) \subseteq NB(S)$. Sincc A is normal $AB = BA$, So $BA \in NB(S)$. Evidently $A(BC) = (AB)C$ for A, B, $C \in NB(S)$. Hence the proposition.

2. 6 Proposition. B (S) is an ideal of NB (S).

Proof. Let $B \in B(S)$ and $X \in NB(S)$. Then $(BX)(BX) = (BXB)X \subseteq BX$ and also $(BX) S (BX) = B(XS) BX \subseteq BX$ So BX is a bi-ideal of S i, e $BX \in B(S)$. Consequently B (S) is a right ideal of NB (S). Similarly we can show that B (S) is also a left ideal of NB (S). Hence B (S) is an ideal of NB (S).

2. 7 Theorem. A semigroup S is normal if and only if NB(S) is normal.

Proof. Let S be a normal semigroup and X, $A \in NB(S)$. Let $a \in A$; then $aX \subseteq aS = Sa \subseteq NB(S)$. a and so $A\, NB(S) \subseteq NB(S)\, A$. Similarly we can prove that the converse inclusion holds. Thus we obtain that A NB (S) = NB (S) A for all A $\in$ NB (S) So NB (S) is normal. Conversely let NB (S) be normal. In order to prove that S is normal, let x $\in$ S. Then for some A $\in$ NB (S) we have $xS \subseteq (x)_B S = A(x)_B \subseteq S(x)_B \subseteq Sx$ where $(x)_B = \{xSx \cup x \cup x^2\}$ is the bi-ideal generated by x and hence $(x)_B \in B(S) \subseteq NB(S)$. Similarly we can prove that the converse inclusion holds. So **S** is normal.

2.8 Theorem. Let S be a normal semigroup ; then the following conditions are equivalent

(1) S is a regular semigroup,

(2) $A \cap B = \bar{B}A$ where A is a left pseudo-ideal and B is a bi-ideal of S ,

(3) $A \cap B = A\bar{B}$ where A is a right pseudo-ideal and B is a bi-ideal of S.

Proof. (1) $\Rightarrow$ (2) . Let S be a normal semigroup which is also regular. Let A, B be respectively a left pseudo-ideal and a bi-ideal of S. Then $\bar{B}A \subseteq A$. Let $b^2 a \in \bar{B}A$ where $b \in B$ and $a \in A$. Since S is normal , bS = Sb. So $b^2 a = bba \in bbS = bSb \subseteq B$. Consequently $\bar{B}A \subseteq B$. Combining the above two inclusions we get $\bar{B}A \subseteq A \cap B$. Conversely, let $c \in A \cap B$. Since S is regular and normal, c = cxc = cxcxcxcxc (since xc is idempotent, $x \in S$) = $cx x_1 c \cdot cx x_1 c \cdot c = (cx x_1 c)^2 c \in \bar{B}A$ ($c \in B$ implies that $cx x_1 c \in B$). Therefore $A \cap B \subseteq \bar{B}A$. Hence $A \cap B = \bar{B}A$.

(2) $\Rightarrow$ (1). Let $c \in S$. Since S is a left pseudo-ideal we have $S \cap (c) = \overline{(c)}\, S$ i, e $(c) = \overline{(c)}\, S$ where (c) denotes the ideal generated by c and hence (c) is a bi-ideal of S. Since S is normal, $(c) = \{c \cup xc : x \in S\}$ Now $c \in (c) = \overline{(c)}\, S$ implies that $c = c^2 y$ or $(xc)^2 z$ where $x, y, z \in S$. Since S is normal, we can write c = cxc for some $x \in S$. So c and hence S is regular.

Similarly we can show that (1) and (3) are equivalent Hence the theorem.

2.9 Lemma. ([6] Corollary II. 4. 12) For an idempotent semigroup S the following conditions are equivalent.

(1) S is normal

(2) S is commutative

2. 10 Lemma. ([1] Theorem 7.6) Following conditions concerning a regular semigroup S are equivalent.

(1) S is normal

(2) es = Se for all idempotents ϵ of S.

2. 11 Lemma. ([4] Theorem 2) For a semigroup S the following conditions are equivalent.

(1) S is a semilattice of groups

(2) S is regular and eS = Se for all idempotents e of S.

2. 12 Lemma, ([2]) For a semigroup S the following conditions are equivalent.

(1) S is regular.

(2) B (S) is regular.

2. 13 Theorem. A normal semigroup S is a semilattice of groups if and only if NB (S) is a semilattice.

Proof : Let S be a normal semigroup which is a semilattice of groups. Ther by lemma 2. 11 S is regular. Let A ϵ NB (S). If A ϵ N (S) and a ϵ A then a = axa = axaxa (since xa is idempotent, x ϵ S) = ax^2a_1 a ϵ AA ($a_1 \epsilon$ A) So A $\subseteq$ AA. On the other hand AA $\subseteq$ A So A = AA Again, if A ϵ B (S) and a ϵ A then a = axa = axaxa= a ax_1xa (since S is normal, xa = ax_1) ϵ AA. So A $\subseteq$ AA. Also AA $\subseteq$ A. Hence for all A ϵ NB (S), A = AA. So NB (S) is idempotent Since S is normal, by theorem 2,7 NB (S) is normal. Hence by lemma 2.9, NB (S) is commutative. Hence NB (S) is a commutative idempotent semigroup i. e, a semilattice. Conversely, let NB (S) be a semilattice. So NB (S) is an idempotent semigroup and hence regular. Since B (S) is an ideal of NB (S), B (S) is also regular. So it follows from lemma 2.12 that S is regular normal semigroup whence by lemma 2.10 and lemma 2.11 it follows that S is a semilattice of groups.

2.14 Theorem. For a normal semigroup S the following conditions are equivalent.

(1) S is regular,

(2) S is left regular,

(3) S is right regular,

(4) S is completely regular,

(5) $a = a^n x^{n-1}$ for all a ϵ S and for every integer n $\geqslant$ 2

(6) NB (S) Is idempotent,

(7) NB (S) is completely regular,

(8) NB (S) is regular,

(9) B (S) is regular.

Proof, It follows from theorem 6.6 of [1] that (1) to (4) are equivalent. Now we assume (4). Let $a \in S$. Then $a=axa$ for some $x \in S$ and $ax=xa$. So $a=a^2x=aax=aa^2xx=a^3x^2$. Continuing this we get $a=a^nx^{n-1}$ for every integer $n\geq 2$

(5) $\Rightarrow$ (6). Let $A \in NB(S)$. If $A \in N(S)$ and $a \in A$ then $a=a^3x^2=a^2 . ax^2 \in AA$. So $A \subseteq AA$ Also $AA \subseteq A$. Thus $A=AA$. If $A \in B(S)$ and $a \in A$ then $a=a^3x^2=a\,a\,ax^2=a\,aya \in AA$ (since S is normal, $ax^2=ya$ for some $y \in S$). So $A \subseteq AA$. Also $AA \subseteq A$. Thus in this case also $A=AA$. So **NB (S) is idempotent.** (6) $\Rightarrow$ (7) $\Rightarrow$ (8) are obvious. Next we assume (8). Since B (S) is an ideal of NB (S), B (S) is regular. Lastly we assume (9). Since B (S) is regular, it follows by lemma 2.12 that S is regular.

A semigroup S is called viable if $ab=ba$ whenever ab and ba are idempotents. A viable semigroup has been studied by M. S. Putcha and J. Weissglass [5]

2.15 Lemma. ([1] Theorem 7.6) For a regular semigroup S the following conditions are equivalent.

(1) S is normal

(2) B (S) is viable

2.16 Lemma. ([5] Theorem 6) If a semigroup S is a semilartice of groups then it is viable.

2.17 Theorem. A regular semigroup S is normal if and only if NB (S) is viable.

Proof Let S be a regular normal semigroup. Then NB (S) is also normal (Theorem 2.7), Also NB (S) is regular (Theeoem 2,14), So NB (S) is regular and normal. Hence **NB (S)** is a semilattice of groups. So by lemma 2.16, NB (S) is viable. Conversely we assume that NB (S) is viable. Since the property of being viableis hereditary, it follows that B (S) is viable whence by lemma 2.15 it follows that S is normal.

2.18 Theorem. The following conditions concerning a semigroup S are equivalent.

(1) S is a semilattice of groups,

(2) S is regular and normal,

(3) NB (S) is regular and normal,

(4) NB (S) is a semilattice of groups,

(5) NB (S) is regular and viable,

(6) NB (S) is a completely regular semigroup and every bi-ideal of S is two-sided.

(7) B (S) Is a completely regular semigroup and every bi-ideal of S is two-sided.

Proof. (1) $\Rightarrow$ (2) follows from the lemma 2.10 and the lemma 2.11. (2) $\Rightarrow$ (3) follows from the proposition 2.7 and the theorem 2.14. (3) $\Rightarrow$ (4) follows from the lemma 2.10 and the lemma 2.11. (4) $\Rightarrow$ (5) follows from the lemma 2.11 and the lemma 2.16. (5) $\Rightarrow$ (6). Since NB (S) is regular and viable it follows readily that NB (S) is a completely regular semigroup. Also since B (S) is an ideal of NB (S) it follows that B (S) is also regular and viable and so every bi-ideal of S is two-sided (Theorem 7.7 of [1]). (6) $\Rightarrow$ (7) follows since **B (S)**. is an ideal of NB (S). Lastly (7) $\Rightarrow$ (1)follows from the theorem 7.7 of [1]. Hence the theorem.

3. In a commutative semigroup or in a group we have noted that every pseudo-ideal is a normal pseudo-ideal. There are also semigroups which are neither commutative semigroups nor groups and in which every pseudo-ideal is normal. In fact the semigroup given in example 2.1 is a semigroup of this type. Now we shall study those semigroups in which every pseudo-ideal is normal. Throughout this article by a semigroup S we shall mean a semigroup in which all the pseudo-ideals are normal.

3.1 Theore. A semigroup S will be regular if and only if $A=\bar{A}A$ where A is a pseudo-ideal of S.

Proof. Let S be a regular semigroup and A be a pseudo-ideal of S. Let $a \in A$, Since S is regular $a=axa$ for some $x \in S$. Now $a=axa=axaxaxaxa$ (since xa is idemotent)$=ax^2 a_1 ax^2a_1 a \in AA$ (since A is normal, $ax=xa_1$ for some $a_1 \in A$) So $A \subseteq \bar{A}A$, Also $\bar{A}A \subseteq A$, Henca $\bar{A}A=A$, Conversely we assume that $A=\bar{A}A$ for all pseudo-ideals A of S. Let $a \in S$ and (a) be the ideal generated by a. Since every ideal is a pseudo-ideal $(a)=\overline{(a)}\,(a)$ Now $a \in (a)=\overline{(a)}\,(a)$ implies $a=axa$ for seme $x \in S$ (Since **S** is normal as **S** is a pseudo-ideal of itself). So a and hence S is regular.

3.2 Theorem. The following conditions concerning a semigroup S are equivalent.

(1) S is regular,

(2) S is left regular,

(3) S is right regular,

(4) S is completely regular,

(5) $a=a^n x^{n-1}$ for every element a ef S and for every integer $n \geqslant 2$.

(6) N(S) is idempotent

(7) N(S) is regnlar

Proof. Since every pseudo-ideal of S is normal S is also normol, so it follows frem the theofem 2.14 that (1) to (5) are equivalent, Now we assume (5). Let $A \in N(S)$ and $a \in A$. Then $a=a^3x^2=a^2. ax^2 \in AA$. So $A \subseteq AA$. Also $AA \subseteq A$. Hence A=AA. So N(S) is idempotent, (6) => (7) is obvious. Lastly we assume (7). Let $a \in S$ and (a) be the ideal generated by a. Since $(a) \in N(S)$ by our assumption there exists A in N(S) such that (a)= (a) A (a) Now a e (a)=(a) A (a) implies that a=axa for some $x \in S$. So a and hence S is regular.

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