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ON THE MAXIMUM TERMS OF THE DERIVATIVES OF ENTIRE FUNCTIONS IN SEVERAL COMPLEX VARIABLES REPRESENTED BY MULTIPLE DIRICHLET SERIES.

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Abstract : A family F of entire functions in several complex variables represented by a multiple Dirichlet series has been considered. Hadamard multiplication and the concept of rank of a maximum term of any function $f_1 \epsilon F$ have been introduced. Partial derivatives of any order of two different functions are considered and a few inequalities involving their maximum terms and ranks have been obtained in R^n, real n-space. After removing the set of discontinuities of the rank from R^n, the special forms and the consequences of the above inequalities are also obtained.

1. **Notations :** We denote complex and real n-space by C^n and R^n respectively and the set of non-negative integers by I, so that I^n will denote the Cartesian product of n copies of I. We indicate the points $(s_1, \ldots, s_n)$, $(\sigma_1, \ldots, \sigma_n)$, $(m_1, \ldots m_n)$ etc. of C^n or R^n by their corresponding unsuffixed symbols s, σ, m respectively and make use of the standard notations of the single variable which are easy to understand from the context.

For s, w ϵ C^n and $\alpha \epsilon C$ where $s = (s_1, \ldots, s_n)$, $w = (w_1, \ldots, w_n)$ we define

i) $s=w$ iff $s_i = w_i$, $i = 1, \ldots, n$

ii) $s+w = (s_1+w_1, \ldots, s_n+w_n)$,

iii) $\alpha s = (\alpha s_1, \ldots, \alpha s_n)$

iv) $s.w = s_1 w_1 + \ldots + s_n w_n$

v) $|s| = \left\{ | s_1 |^2 + \ldots + | s_n |^2 \right\}^{1/2}$

For $a \epsilon R$, $s \epsilon C^n$,

vi) $s+a = (s_1+a, \ldots\ldots s_n+a)$

Also for $x, y \in R^n$, we say that

vii) $x \leq y$ iff $x_i \leq y_i$, $i = 1, \ldots\ldots, n$

viii) $x < y$ iff $x \leq y$ but $x \neq y$

ix) $x << y$ iff $x_i < y_i$, $i = 1, \ldots\ldots, n$.

The positive hyperoctant R^n_+ in R^n will be

$R^n_+ = \{x : x \in R^n, x_i \geq 0, i = 1, \ldots\ldots, n\}$. For $t \in R^n_+$

we set $\|t\| = t_1 + \ldots\ldots + t_n$ and for $m \in I^n$, $m! = m_1! \ldots m_n!$.

Also for $s \in C^n$, $t \in R^n_+$ we shall denote $s_1^{t_1} \ldots s_n^{t_n}$ by

s^t, ($s_i^0 = 1$ even if $s_i = 0$). For $k \in R$, $\bar{k}$ will denote the real n-tuple ($k, \ldots\ldots, k$). For an entire function f with domain C^n, f^k will denote the function $\dfrac{\partial^{\|k\|} f}{\partial s_1^{k_1} \ldots \partial s_n^{k_n}}$ where $k \in I^n$ and

$f^{\bar{0}} = f$.

An unorthodox notation : In the definition of a multiple Dirichlet series we take n sequences $\left\{\lambda_{jm_j}\right\}_{m_j=1}^{\infty}$, $j = 1, \ldots\ldots n$ of exponents.

We shall often require the n-tuple $(\lambda_{1m_1}, \ldots, \lambda_{nm_n})$ of those sequences. For brevity we denote this n-tuple by (λ_{nm_n}). Also, for a particular set of values of $m_1, \ldots\ldots m_n$, say $p = (p_1, \ldots\ldots p_n)$, the n-tuple $(\lambda_{1p_1}, \ldots\ldots, \lambda_{np_n})$ will be denoted by (λ_{np}). Thus, $s.(\lambda_{nm_n})$ will mean $s_1\lambda_{1m_1} + \ldots + s_n\lambda_{nm_n}$.

2. We consider an entire function in C^n represented by the multiple Dirichlet series

$$f_1(s_1, \ldots\ldots s_n) = \sum_{m_1, \ldots\ldots, m_n = 1}^{\infty} a_{m_1 \ldots\ldots m_n} \exp(s_1\lambda_{1m_1} + \ldots\ldots + s_n\lambda_{nm_n})$$

i. e.

$$(2.\ 1.) \quad f_1(s) = \sum_{m=1}^{\infty} a_m \exp\{ s.(\lambda_{nm_n})\}, \text{ where}$$

$s_j = \sigma_j + i\tau_j \in C$ ($j = 1, \ldots\ldots n$), $a_m \in C$ and $\left\{ \lambda_{j m_j} \right\}_{m_j = 1}^{\infty}$

are n sequences of exponents satisfying the conditions

(2. 2) $0 < \lambda_{j1} < \lambda_{j2} < \cdots < \lambda_{jk} \to \infty$ as $k \to \infty$, for $j = 1, \ldots\ldots, n$. Throughout we tacitly assume that

(2. 3) $$\lim_{m_j \to \infty} \frac{\log m_j}{\lambda_{j m_j}} = 0, \ j = 1, \ldots\ldots, n.$$

A. I. Janusauskas [2] had shown that if (2. 3) holds then the domain of convergence of the series (2. 1) coincides with its domain of absolute convergence.

Let F be the family of all entire functions represented by a series of the form

(2.1) having the same sequences of exponents $\left\{ \lambda_{j m_j} \right\}_{m_j = 1}^{\infty}$, ($j = 1, \ldots\ldots, n$)

and satisfying the condition (2.2).

For $f_1, f_2 \in F$, we define $f = f_1 * f_2$ by

(2. 4) $f(s) = f_1(s) * f_2(s) = \sum_{m=1}^{\infty} a_m b_m \exp\left\{ s.\ (\lambda_{n m_n}) \right\}$ where

(2. 5) $f_1(s) = \sum_{m=1}^{\infty} a_m \exp\left\{ s.\ (\lambda_{n m_n}) \right\}$ and

(2. 6) $f_2(s) = \sum_{m=1}^{\infty} b_m \exp\left\{ s.\ (\lambda_{n m_n}) \right\}$.

Throughout this paper f will denote the product function $f_1 * f_2$ as given in (2.4).

Theorem 1. The function f, as defined by (2.4), belongs to F.

Proof : Since (2.5) is entire, the series $\sum_{m=1}^{\infty} | a_m | \exp\left\{ \sigma.\ (\lambda_{n m_n}) \right\}$ is convergent

for all $\sigma \in R^n$. In particular, it is convergent at $\sigma = \bar{0}$, so that $\sum_{m=1}^{\infty} | a_m |$ is

convergent. Thus, $\lim_{\| m \| \to \infty} | a_m | = 0$ and hence the n-sequence $\left\{ | a_m | \right\}$ is

bounded. Also the series $\sum_{m=1}^{\infty} | b_m | \exp\left\{ \sigma.\ (\lambda_{n m_n}) \right\}$ is convergent for all $\sigma \in R^n$

and consequently $\sum_{m=1}^{\infty} |a_m|\,|b_m| \exp\left\{\sigma.(\lambda_{nm_n})\right\}$ is convergent for all $\sigma \in R^n$

which implies $\sum_{m=1}^{\infty} a_m b_m \exp\left\{s.(\lambda_{nm_n})\right\}$ is absolutely convergent for all $s \in C^n$.

Hence (2.4) represents an entire function and $f \in F$.

3. Corresponding to any $f_1 \in F$ we define the functions : the maximum modulus $M(\sigma, f_1)$, the maximum term $\mu(\sigma, f_1)$ and the indices $\nu_j(\sigma, f_1)$ of the maximum term $\mu(\sigma, f_1)$, (j = 1,...... n) on R^n by $M(\sigma, f_1) = \max_{\text{Re } s = \sigma} |f(s)|$,

$$\mu(\sigma, f_1) = \max_{m \in I^n} [\,|a_m| \exp\left\{\sigma.(\lambda_{nm_n})\right\}]$$

$$\nu_j(\sigma, f_1) = \max [\,m_j : |a_m| \exp\left\{\sigma.\lambda_{nm_n}\right\} = \mu(\sigma, f_1)],\ j = 1,\dots\dots,n.$$

We call $\nu = \nu(\sigma, f_1) = (\nu_1,\dots\dots\nu_n)$ as the rank of the maximum term $\mu(\sigma, f_1)$. It is shown in theorem I that the series (2.4) belongs to F. Consequently, for $k \in I^n$,

$$f^k(s) = (f_1(s) * f_2(s))^k = \sum_{m=1}^{\infty} (\lambda_{nm_n})^k a_m b_m \exp\left\{s.\lambda_{nm_n}\right\} \text{ and } \overset{k}{\underset{*}{f}}(s) = \overset{k}{f}_1(s) * \overset{k}{f}_2(s)$$

$$= \sum_{m=1}^{\infty} (\lambda_{nm_n})^{2k} a_m b_m \exp\left\{s.(\lambda_{nm_n})\right\} \text{ are also elements of F.}$$

Let $M(\sigma, k)$, $M^*(\sigma, k)$ be the maximum modulii of f^k and $\overset{k}{\underset{*}{f}}$ respectively and their respective maximum terms be denoted by $\mu(\sigma, k)$ and $\mu^*(\sigma, k)$. Then,

$$\mu(\sigma, k) = \max_{m \in I^n} [\,(\lambda_{nm_n})^k |a_m b_m| \exp\left\{\sigma.(\lambda_{nm_n})\right\}],$$

$$\mu^*(\sigma, k) = \max_{m \in I^n} [\,(\lambda_{nm_n})^{2k} |a_m b_m| \exp\left\{\sigma.(\lambda_{nm_n})\right\}].$$

Also let $\nu_j = \nu_j(\sigma, k)$, and $\nu_j^* = \nu_j^*(\sigma, k)$, j = 1, ..., n, be the indices of $\mu(\sigma, k)$ and $\mu^*(\sigma, k)$ respectively and $\nu = \nu(\sigma, k) = (\nu_1, \dots\dots, \nu_n)$, $\nu^* = \nu^*(\sigma, k) = (\nu_1^*, \dots\dots, \nu_n^*)$ be their respective ranks.

Theorem 2. For $\bar{0} \leq \sigma << \zeta$ and $k \in I^n$,

$$M(\sigma, k) \leqslant \frac{k!\, M^*(\zeta, \bar{0})}{(\zeta - \sigma)^k}$$

Proof : By Cauchy's integral formula

$$\frac{\partial^{\|k\|} f(s)}{\partial s_1^{k_1} \ldots\ldots \partial s_n^{k_n}} = \frac{k!}{(2\pi i)^n} \int_C \frac{f(w)}{(w-s)^{k+1}} \, dw_1 \ldots\ldots dw_n ,$$

where $C = C_1 \times \ldots \times C_n$, $C_i : | w_i - s_i | = \xi_i - \sigma_i$, $i = 1, \ldots\ldots, n$.

Hence, $$\left| \frac{\partial^{\|k\|} f(s)}{\partial s_1^{k_1} \ldots\ldots \partial s_n^{k_n}} \right| \leq \frac{k!\; M(\zeta, \bar{O})}{(\zeta - \sigma)^k}$$

Since $M(\sigma, \bar{O}) = M^*(\sigma, \bar{O})$ for all $\sigma \in R^n$, the theorem follows.

Theorem 3. For any $\sigma \in R^n$ and $k \in I^n$,

$$(\lambda_{np})^k \leqslant \lambda \frac{\mu^*(\sigma, k)}{\mu(\sigma, k)} \leqslant (\lambda_{n\nu^*})^k ,$$

where $p = (p_1, \ldots\ldots, p_n)$ is a position of occurrence of $\mu(\sigma, k)$ and $\nu^*(\sigma, k)$ is th rank of $\mu^*(\sigma, k)$.

Proof : Let $\mu(\sigma, k)$ occur at a position $p = (p_1, \ldots\ldots, p_n)$ and $\mu^*(\sigma, k)$ occur a $p^* = (\overset{*}{p_1}, \ldots\ldots \overset{*}{p_n})$. Then,

$$\mu(\sigma, k) = (\lambda_{np})^k \, | a_p b_p | \; \exp \left\{ \sigma.(\lambda_{np}) \right\}$$

$$\geqslant (\lambda_{np*})^k \; | a_{p*} b_{p*} | \; \exp \left\{ \sigma. (\lambda_{np*}) \right\} = \frac{\mu^*(\sigma, k)}{(\lambda_{np*})^k}$$

Hence, $\dfrac{\mu^*(\sigma, k)}{\mu(\sigma, k)} \leq (\lambda_{np*})^k$, Evidently $p* \leq \nu^*$. Due to (2.2) we have,

$$(3.1) \qquad \frac{\mu^*(\sigma, k)}{\mu(\sigma, k)} \leqslant (\lambda_{np}{}^*)^k.$$

Again, $\mu^*(\sigma, k) = (\lambda_{np*})^{2k} \, | a_{p*} b_p{}^* | \exp \left\{ \sigma. (\lambda_{np*}) \right\}$

$\geqslant (\lambda_{np})^{2k} \, | a_p b_p | \; \exp \{ \sigma. \lambda_{np} \} = (\lambda_{np})^k \, \mu(\sigma, k)$. Hence,

$$(3.2) \qquad (\lambda_{np})^k \leq \frac{\mu^*(\sigma, k)}{\mu(\sigma, k)}$$

Combining (3.1) and (3.2) the theorem follows.

Theorem 4 : For any $k >> \bar{O}$ and $\sigma \in R^n$,

$$\lambda_{1q_1} \cdots\cdots \lambda_{nq_n} \leqslant \frac{\mu(\sigma, k)}{\mu(\sigma, k\text{-}l)} \leqslant \lambda_1\nu_1 \cdots\cdots \lambda_n\nu_n,$$

where $q = (q_1, \ldots\ldots, q_n)$ is a position of occurrence of $\mu(\sigma, k\text{-}l)$ and $\nu =$ $(\nu_1, \ldots\ldots, \nu_n)$ is the rank of $\mu(\sigma, k)$,

Proof : Let $\mu(\sigma, K)$ occur at $p = (p_1, \ldots\ldots, p_n)$. Then,

$$\mu(\sigma, k\text{-}l) = (\lambda_{nq})^{k-1} \mid a_q b_q \mid \exp\{\sigma \cdot (\lambda_{nq})\}$$

$$\geqslant (\lambda_{np})^{k-1} \mid a_p b_p \mid \exp\{\sigma . (\lambda_{nj})\} = \frac{\mu(\sigma, k)}{\lambda_{1p_1} \cdots\cdots \lambda_{np_n}}, \quad \text{Hence,}$$

(3. 3) $$\frac{\mu(\sigma, k)}{\mu(\sigma, k\text{-}l)} \leqslant \lambda_{1p1} \cdots \lambda_{npn} \leqslant \lambda_1\nu_1 \cdots \lambda_n\nu_n.$$

Again $\mu(\sigma, k) \geqslant (\lambda_{nq})^k \mid a_q b_q \mid \exp\{\sigma . (\lambda_{nq})\}$

$= \lambda_{1q1} \cdots \lambda_{nq_n} \mu(\sigma, k\text{-}l)$. Hence,

(3. 4) $\lambda_{1q1} \cdots \lambda_{nqn} \leqslant \dfrac{\mu(\sigma, k)}{\mu\, \sigma(, k\text{-}l)}$; combining (3. 3) and (3. 4) the theorem follows.

Theorem 5. For any $k \geqslant \bar{O}$ and $\sigma \in R^n$,

$$\lambda 1_{q_1}{}^* \ldots \lambda_{nqn}{}^* \leqslant \left\{ \frac{\mu^*(\sigma, k)}{\mu^*(\sigma, k\text{-}l)} \right\}^{\frac{1}{2}} \leqslant \lambda_1 \nu_1{}^* \cdots \lambda_n \nu_n{}^*,$$

where. $q^* = (\overset{*}{q_1}, \ldots\ldots, \overset{*}{q_n})$ is a position of occurrence of $\mu^*(\sigma, k\text{-}l)$ and $\nu^* =$ $(\nu_1{}^*, \ldots\ldots, \nu_n{}^*)$ is the rank of $\mu^*(\sigma, k)$.

Proof : The proof is exactly similar to that of theorem 4.

4. For any $f_1 \in F$, let D be the set of all discontinuities of ν in R^n, where $\nu = (\nu_1, \ldots\ldots \nu_n)$ is the rank of the maximum term $\mu(\sigma, f_1)$. Also let S denote the set of all $\sigma \in R^n$ at which $\mu(\sigma, f_1)$ is attained by more than one term of the series

(4. 1) $$\sum_{m=l}^{\infty} \mid a_m \mid \exp\left\{ \sigma . (\lambda_{nm_n}) \right\}.$$

R. K. Das [1] had shown that D and S are identical. (A similar result for entire functions represented by multiple power series was proved by J. G. Krishna [3]).

Hence for $\sigma \in R^n$-D, $\mu(\sigma, f_1)$ is attained by only one term of the series (4. 1) and the position of that term is $\nu = (\nu_1, \ldots, \nu_n)$.

Hence in such a case, the theorems 3,4 and 5 will take the following forms in theorems 6, 7 and 8 respectively.

Theorem 6. For any $\sigma \in R^n - D$ and $k \in I^n$,

$$(4.2)\quad (\lambda_n \nu) \leq \frac{\mu^*(\sigma, k)}{\mu(\sigma, k)} \leq (\lambda_n \nu^*)$$

where ν and ν^* are the ranks of μ and μ^* respectively and D is the set of all discontinuities of $\nu(\sigma, k)$ in R^n.

Theorem 7. For any $\sigma \in R^n - D_1$ and $k >> \bar{0}$,

$$(4.3)\quad \lambda_1 \nu_1(\sigma, k\text{-}I) \dots \lambda_n \nu_n(\sigma, k\text{-}I) \leq \frac{\mu(\sigma, k)}{\mu(\sigma, k\text{-}I)} \leq \lambda_1 \nu_1(\sigma, k) \dots \lambda_n \nu_n(\sigma, k),$$

where D_1 is the set of all discontinuities of $\nu(\sigma, k\text{-}I)$.

Corollaries : For any $\sigma \in R^n\text{-}D_1$,

i) $\lambda_1 \nu_1(\sigma, \bar{0}) \dots \lambda_n \nu_n(\sigma, \bar{0}) \leq \lambda_1 \nu_1(\sigma, \bar{1}) \dots \lambda_n \nu_n(\sigma, \bar{1}) \dots) \leq \dots$

ii) $\dfrac{\mu(\sigma, \bar{1})}{\mu(\sigma, \bar{0})} \leq \dfrac{\mu(\sigma, \bar{2})}{\mu(\sigma, \bar{1})} \leq \dots\dots$

iii) Putting $k = \bar{1}, \bar{2}, \dots\dots, \bar{p}$ successively in (4.3) and multiplying the resulting p inequalities and using (i) we have,

$$\lambda_1 \nu_1(\sigma, \bar{0}) \dots \lambda_n \nu_n(\sigma, \bar{0}) \leq \left\{\frac{\mu(\sigma, \bar{p})}{\mu(\sigma, \bar{0})}\right\}^{\frac{1}{p}} \leq \lambda_1 \nu_1(\sigma, \bar{p}) \dots\dots \lambda_n \nu_n(\sigma, \bar{p}).$$

Theorem 8. For any $\sigma \in R^n - D_1^*$ and $k \geq \bar{0}$

$$(4.4)\quad \lambda_1 \nu_1^*(\sigma, k\text{-}I) \cdots \lambda_n \nu_n^*(\sigma, k\text{-}I) \leq \left\{\frac{\mu^*(\sigma, k)}{\mu^*(\sigma, k\text{-}I)}\right\}^{1/2} \leq \lambda_1 \nu_1^*(\sigma, k) \cdots \lambda_n \nu_n^*(\sigma, k)$$

where D_1^* is the set of all discontinuities of $\nu^*(\sigma, k\text{-}I)$.

Corollaries : For any $\sigma \in R^n\text{-}D_1^*$

i) $\lambda_1 \nu_1^*(\sigma, \bar{0}) \cdots \lambda_n \nu_n^*(\sigma, \bar{0}) \leq \lambda_1 \nu_1^*(\sigma, \bar{1}) \cdots \lambda_n \nu_n^*(\sigma, \bar{1}) \leq \dots$

ii) $\dfrac{\mu^*(\sigma, \bar{1})}{\mu^*(\sigma, \bar{0})} \leq \dfrac{\mu^*(\sigma, \bar{2})}{\mu^*(\sigma, \bar{1})} \leq \dots$

iii)

$$\lambda_1 \nu_1^*(\sigma, \bar{O}) \cdots \lambda_n \nu_n^*(\sigma, \bar{O}) \leqslant \left\{ \frac{\mu^*(\sigma, \bar{p})}{\mu^*(\sigma, \bar{O})} \right\}^{\frac{1}{2p}} \leqslant \lambda_1 \nu_1^*(\sigma, \bar{p}) \cdots$$

$$\lambda_n \nu_n^*(\sigma, \bar{p})$$

Theorem 9. For any $\sigma \in R^n - D \cup D_1$ and $k \in I\ (k > O)$,

$$\lambda_1 \nu_1(\sigma, \overline{k\text{-}1}) \cdots \lambda_n \nu_n(\sigma, \overline{k\text{-}1}) \leqslant \left\{ \frac{\mu^*(\sigma, \bar{k})}{\mu(\sigma, \overline{k\text{-}1})} \right\}^{\frac{1}{k+1}} \leqslant \lambda_1 \nu_1^*(\sigma, k)$$

$$\ldots \lambda_n \nu_n^*(\sigma, \bar{k}),$$

where D is the set of all dicontinuities of $\nu(\sigma, \bar{k})$ and D_1 is the set of ali disconti-nuities of $\nu(\sigma, \overline{k\text{-}1})$.

Proof : From (4.2), using (4.3), we have

$$\mu^*(\sigma, \bar{k}) \leqslant \mu(\sigma, \bar{k}) (\lambda_n \nu^*(\sigma, \bar{k}))^{\bar{k}}$$

$$\leq \mu(\sigma, \overline{k\text{-}1}) \lambda_1 \nu_1(\sigma, \bar{k}) \cdots \lambda_n \nu^*(\sigma, \bar{k}) (\lambda_n \nu_n^*(\sigma, \bar{k}))$$

Hence,

$$(4.5) \quad \frac{\mu^*(\sigma, \bar{k})}{\mu(\sigma, \overline{k\text{-}1})} \leqslant \lambda_1 \nu_1(\sigma, \bar{k}) \cdots \lambda_n \nu_n(\sigma, \bar{k}) (\lambda_n \nu^*(\sigma, \bar{k}))^{\bar{k}}$$

Again, from (4. 2), $\mu^*(\sigma, \bar{k}) \geqslant \mu(\sigma, \bar{k}) (\lambda_n \nu(\sigma, \bar{k}))^{\bar{k}} \geqslant \mu(\sigma, \overline{k\text{-}1})^{\times}$

$$\lambda_1 \nu_1(\sigma, \overline{k\text{-}1}) \ldots \lambda_n \nu_n(\sigma, \overline{k\text{-}1}) (\lambda_n \nu(\sigma, \bar{k}))^{\bar{k}}.$$

Hence,

$$(4.6) \quad \frac{\mu^*(\sigma, \bar{k})}{\mu(\sigma, \overline{k\text{-}1})} \geqslant \lambda_1 \nu_1(\sigma, \overline{k\text{-}1}) \cdots \lambda_n \nu_n(\sigma, \overline{k\text{-}1}) (\lambda_n \nu(\sigma, \bar{k}))^{\bar{k}}.$$

From (4.5) and (4.6), using (4.2) and the Corollary (i) of theorem 7, the result follows. The second author wishes to express his gratitude to the "Govt. of India" for awarding him a scholarship.

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