Jour. Pure Math.
Vol. 5 (1985—86), pp. 15–26

ON COMPARISONS OF CERTAIN NON-CONTINUOUS MULTIVALUED FUNCTIONS BETWEEN BITOPOLOGICAL SPACES

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Abstract

In this paper certain classes of multivalued functions strictly weaker than continuous one, have been introduced for bitopological spaces, which present extended and generalized versions of their single-valued and multivalued counterparts between topological spaces. They have been characterized and studied specially with regard to their mutual dependence and interrelations.

1. Introduction

For the last quarter of a century, different mathematicians have been taking keen interest in the introduction and study of numerous kinds of mappings in topological spaces, most of which are strictly weaker than the usual continuous or open maps. Such a vast study has not only effectively characterized various concepts of topology but altogether new directions of further research and study have emerged. Some of these maps have been generalized to their multivalued forms too.

The notion of weak continuous map on topological spaces was first introduced and studied by N. Levine [5] followed by its further study by T. Noiri [10. 11, 12] and others. M. K. Singal and A. R. Singal [13] introduced the concept of a very important class of a non-continuous map which they termed almost continuous function. This kind of map was later found to be a natural tool and extremely useful for studying nearly compact spaces, almost regular spaces and for fruitfully characterizing H-closed spaces as the almost continuous images of minimal Hausdorff spaces. Due to its effectiveness and use in application, the concept was subsequently generalized to fuzzy topological situation by Azad [1] and to its multivalued form in a more generalized structure of bitopological spaces in [9] Functions between topological spaces under the same terminology viz. almost continuity were also studied by Husain [3] and others [2, 15], but each of those functions is independent of that of Singal and Singal. Investigations of these functions along with their mutual interactions are found in [6, 7, 8]. A certain study of almost continuous (in the sense of Husain) and weakly continuous multifunctions is done by Smithson [14], generalizing some results derived earlier for single-valued case. After the introduction of the theory of bitopological structures by J. C. Kelly [4] in 1963, the last two becades have witnessed a tremendous growth of the theory resulting to a vast literature of papers dealing with numerous concepts of topology in more generalized premises in a very effective manner. Such structures are seen to be naturally inherent in certain situations like quasiuniform, quasi-pseudometric or quasi proximity space and the theory contains the theory of topological spaces in particular. Apart from extension of concepts of topology to a more generalized perspective, the study of bitopological spaces has alreaby shown some real worth in getting newer concepts, more general, more fruitful, specially when the two topologies are very much naturally associated. Though a good number of papers have appeared dealing with some single-valued maps between bitopological spaces, the multifunctions have recently been touched demanding a substantive theory in this context to be evolved.

With the above motivation in view, our aim in this paper is to introduce and study the multivalued forms of weak continuity of Levine and almost continuity of Husain in bitopological spaces and make a conparative study of these maps along with almost continuous multifunctions studied in [9].

By X and Y we shall always mean the bitopological spaces (X, P_1, P_2) and (X, Q_1, Q_2) where P_1, P_2 (Q_1, Q_2) are two arbitrary topologies on X (respectively Y) and F will denote a multifunction from X to Y. P_i-clA and P_i-intA will respectively stand for closure and interior of a subset A of X with respect to the topology P_i on X, for i = 1 or 2. Similarly the notations Q_i-clB and Q_i-intB are defined. We make the convention that in any sentence where the suffixes i & j both appear it is understood that i, j = 1, 2 and $i \neq j$.

2. Weakly Continuous Multifunctions Between Bitopological Spaces

Definition 2.1 Let (X, P_1, P_2) aud (Y, Q_1, Q_2) be two bitopotogical spaces and $F: X \rightarrow Y$ be a multifunction. Then

(a) the upper inverse $F^+(G)$ and lower inverse $F^-(G)$ of a subset G of Y under F are defined by

$$F^+(G) = \{x \in X : F(x) \subset G\}, \quad F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\},$$

(b) The multivalued graph function G_F of F is defined to be the function from (X, P_1, P_2) to $(X \times Y, P_1 \times Q_1, P_2 \times Q_2)$ given by $G_F(x) = \{(x, y) : y \in F(x)\}$ for $x \in X$; by R_i we shall denote the product topology $P_i \times Q_i$ (for i = 1, 2) on the product space $X \times Y$.

Definition 2.2 Let $F : (X, P_1, P_2) \rightarrow (Y, Q_1, Q_2)$ be a multifunction.

(a) F is called Q_i (P_iQ_j)-upper weakly continuous (u.w.c.-in short) if for each point $x_0 \in X$ and each Q_i-open set V with $F(x_0) \subset V$, there exists a P_i-open neighbourhood (henceforth nbd., in short) U of x_0 such that $F(x) \subset Q_j\text{-cl}V$, for all $x \in U$, i, j=1, 2 and $i \neq j$.

(b) F is called Q_i (P_iQ_j)-lower weakly continuous (l.w c.-in short) if for each point $x_0 \in X$ and each Q_i-open set V with $F(x_0) \cap V \neq \emptyset$, there exists a P_i-open nbd. U of x_0 such that $F(x) \cap Q_j\text{-cl } V \neq \emptyset$, for every $x \in U$ (i.j=1,2 and $i \neq j$).

(c) F is called pairwise u.w.c. (pairwise l.w.c) if F is $Q_1(P_1Q_2)$-u.w.c. (l.w.c.) and $Q_2(P_2Q_1)$-u.w.c (l. w. c.).

(d) F is called pairwise weakly continuous if F is pairwise u,w.c. as well as pairwise l.w.c.

Theorem 2.3 A multifunction $F : X \rightarrow Y$ is $Q_i(P_iQ_j)$-u.w.c. iff its graph function $G_F : X \rightarrow (X \times Y, R_1, R_2)$, where $R_i = P_i \times Q_i$ (for i = 1, 2), is $R_i(P_iR_j)$-u.w.c.,for i,j=1,2 and $i \neq j$.

Proof Suppose F is Q_i (P_iQ_j)-u.w.c. and $x_0 \in X$ be arbitrary. Let W be a R_i-open set with $G_F(x_0) \subset W$. Then $G_F(x_0) \subset U \times V \subset W$, where $U \in P_i$ and $V \in Q_i$, so that $F(x_0) \subset V \in Q_i$. Since F is Q_i (P_iQ_j)-u.w.c , there exists a p_i-open nbd. U' of x_0 with $U' \subset U$ such that $F(U') \subset Q_j\text{-cl}V$. Then $G_F(U') = U' \times F(U') \subset U \times Q_j\text{-cl}V \subset R_j\text{-cl }(U \times V) \subset R_j\text{cl } W$. Hence G_F is $R_i(P_iR_j)$-u.s.c. conversely, let G_F be $R_i(P_iR_j)$-u.s.c. and let $x_0 \in X$ be arbitrary. If $V \in Q_i$ with $F(x_0) \subset V$, then $G\ (x_0) \subset X \times V \in R_i$ and hence by hypothesis, there exists a P_i-open nbd. U of x_0 such that $G_F(U) \subset R_j\text{-cl }(X \times V) = X \times Q_i\text{-cl } V$. That means $U \times F(U) \subset X \times Q_j\text{-cl } V$ so that $F(U) \subset Q_j\text{-cl } V$ and hence F is Q_i (P_iQ_j)-u.w.c.

Corollary 2.4 $F : X \rightarrow Y$ is pairwise u.w.c. iff its graph function G_F is so.

Theorem 2.5 A multifunction $F : X \rightarrow Y$ is Q_i (P_iQ_j)-l.w.c. iff its graph function $G_F : X \rightarrow (X \times Y, R_1, R_2)$ is R_i (P_iR_j)-l.w.c.. for i, j=1, 2 and $i \neq j$.

Proof. Let F be Q_i (P_iQ_j)-l.w.c. and $x_0 \in X$ be arbitrary. If $W \in R_i$ with $G_F(x_0) \cap W \neq \emptyset$, then there exist $U \in P_i$, $V \in Q_i$ such that $U \times V \subset W$ and $F(x_o) \cap V \neq \emptyset$. By hypothesis, there exists a P_i-open nbd. U of x_0 such that $F(x)\cap Q_j$-cl V $\neq \emptyset$, for all $x\in$ U. Now, $G_F(x)\cap R_j$-cl $(U\times V)=[\{x\}\times F(x)]\cap R_j$-cl $(U\times V)=[\{x\}\times F(x)]\cap[P_j$-cl$U\times Q_j$-cl$V]\neq\emptyset$. for all $x\in U$ so that $G_F(x)\cap R_j$-cl W $\neq\emptyset$, for all $x\in U$ and hence G_F is R_i (P_iR_j)-l.w.c. Conversely; let $V\in Q_i$ such that $F(x_0)\cap V\neq\emptyset$. Now $G_F(x_0)\cap(X\times V)\neq\emptyset$, where $X\times V\in R_i$. Since G_F is R_i (P_iR_j) l.w.c. there is P_i-open nbd. U of x_0 such that $G_F(x)\cap R_j$-cl $(X\times V)\neq\emptyset$, for all $x\in U$, i.e., $[\{x\}\times F(x)]\cap[P_j$-cl $X\times Q_j$-cl V$]\neq\emptyset$, so that $F(x)\cap Q_j$-cl V $\neq\emptyset$, for all $x\in$ U. Hence F is Q_i (P_iQ_j)-l.w.c.

Corollary 2.6 $F: X\to Y$ is pairwise l.w.c. iff its graph function G_F is so.

From Corollaries 2.4 and 2.6 we obtain—

Corollary 2.7 A multifunction $F: X\to Y$ is pairwise weakly continuous iff the multi-valued graph function G_F of F is pairwise weakly continuous.

Theorem 2.8 If a multifunction $F: X\to Y$ is Q_i (P_iQ_j)-u.w.c. then P_i-cl $[F^-(V)]\subset F^-$ $(Q_i$-cl V), for every Q_j-open set V.

Proof. Suppose $x\notin F^-(Q_i$-cl V), where $V\in Q_j$. Then $F(x)\subset Y-Q_i$-cl$V\in Q_i$. Since F is Q_i (P_iQ_j)-u.w.c.. there exists a P_i-open nbd. U of x such that $F(U)\subset Q_j$-cl $(Y-Q_i$-cl V). Then $F(U)\cap V=\emptyset$, since V is Q_j-open so that $U\cap F^-(V)=\emptyset$. Hence $x\notin P_i$-cl $[F^-(V)]$.

Theorem 2.9 If a multifunction $F: X\to Y$ is $Q_i(P_iQ_j)$-l.w.c., then $P_i$cl$[F^+(V)]\subset F^+$ $(Q_i$cl V), for every Q_j-open set V.

Proof. Let $x\notin F^+(Q_i$-clV), where $V\in Q_j$. Then $F(x)\not\subset Q_i$-cl V so that $F(x)\cap(Y-Q_i$-clV) $\neq\emptyset$. Since F is Q_i (P_iQ_j)-l.w.c., there exists a P_i-open nbd. U of x such that $F(x')\cap Q_j$-cl $(Y-Q_i$-cl V) $\neq\emptyset$, for all $x'\in U$. Then $F(x')\not\subset V$, for all $x'\in U$, since $V\cap Q_j$-cl$(Y-Q_i$-clV$)=\emptyset$. Then $U\cap F^+(V)=\emptyset$, where $x\in U\in P_i$. Hence $x\notin P_i$-cl $F^+(V)$.

We know that the continuity of multifunctions between topological spaces is defined by the introduction of two associated concepts viz. lower semi-continuity and upper semi-continuiy. Analogously we define the notion of pairwise continuity of multifunctions between bitopological spaces as follows.

Definition 2.10 A multifunction $F: X\to Y$ is said to be

(a) $Q_i(P_i)$-lower semi-continuous (l.s.c.-in short) if for each point x_0 of X and every Q_i-open set V in Y with $F(x_0)\cap V\neq\emptyset$, there is a P_i-open nbd. U of x_0 such that $F(x)\cap V\neq\emptyset$, for all x of U (i=1, 2).

(b) $Q_i(P_i)$-upper semi-continuous (u.s.c -in short) if for each point x_0 of X and every Q_i-open set V in Y with $F(x_0)\subset Y$, there is a P_i-open nbd. U of x_0 such that $F(U)\subset V$ (i=1, 2),

(c) pairwise l.s.c. (u.s.c.) if F is $Q_1(P_1)$-l.s.c. (u.s.c.) as well as $Q_2(P_2)$-l.s.c. (u.s.c.),

(d) pairwise continuous if F is pairwise l.s.c. and pairwise u.s.c.

It is clear that every pairwise l.s.c. (u.s.c., continuous) multifunction is pairwise l.w.c. (respectively u.w.c., weakly continuous). In order to investigate for the converse problem we require the following definitions.

Definition 2 11 [4] A space (X, P_1, P_2) is called P_i (P_j)-regular if for each x in X and each P_i-closed set V with $x \notin V$, there is a P_i-open set U and a P_j-open set W disjoint from U such that $x \in U$ and $V \subset W$, where, as before, $i, j = 1, 2$ and $i \neq j$, X is called pairwise regular iff it is P_1 (P_2)-regular and P_2 (P_1)-regular.

Definition 2.12 A set A of a bitopological space (X, P_1, P_2) is called strictly P_i (P_j) -paracompact iff every cover $\mathcal{Y}$ of A with P_i-open sets has a refinement $\mathcal{B}$ with P_i-open sets, which cover A and $\mathcal{B}$ is P_j, locally finite, i,e., for each point x of X there is a p_j-open nbd. U of x intersecting at most finitely many elements of $\mathcal{B}$. A is called strictly pairwise paracompact if it is strictly P_1 (P_2)-as well as P_2 (P_1)-paracompact.

Theorem 2.13 If a multifunction $F : X \rightarrow Y$ is Q_i $(P_i$ $Q_j)$-l.w.c. and Y is Q_i (Q_j)-regular, then F is Q_i (P_i)-l.s,c.

Proof : Let $x_0 \in X$ be arbitrary and V be a Q_i-open set with $F(x_0) \cap V \neq \emptyset$. Let $y \in F(x_0) \cap V$. Since Y is Q_i (Q_j)-regular and $y \in V \in Q_i$, there exists $D \in Q_i$ such that $y \in D \subset Q_j\text{-cl}\, D \subset V$. Now since $D \in Q_i$ and $y \in F(x_0) \cap D$, there exists P_i-open nbd. U of x_0 such that $F(x) \cap Q_j\text{-cl}\, D \neq \emptyset$, for all $x \in U$. Then $F(x) \cap V \neq \emptyset$, for all $x \in U$ and hence F is Q_i $(P_i$)-l. s. .c.

Corollary 2.14 For a multifunction F from a bitopological space to a pairwise regular space the concepts of pairwise lower weak continuity and pairwise lower semi-continuity coincide.

Theorem 2.15 Let $(Y, Q_1 . Q_2)$ be Q_i (Q_j)—regular and for each $x \in X$, F (x) is strictly Q_i (Q_j)-paracompact, where $F : X \rightarrow Y$ is Q_i $(P_i$ $Q_j)$- u.w.c. Then F is Q_i (P_i)-u.s.c.

Proof. Being similar to that of Theorem 3.9 of [9] is omitted.

Corollary 2.16 For a multifunction F from a bitopological space X to a pairwise regular space the concepts of pairwise upper weak continuity and pairwise upper semicontinuity coincide, provi ed F (x) is strictly pairwise paracompact for each x of X.

3. WEAKLY CONTINUOUS AND S. ALMOST CONTINUOUS MULTIFUNCTIONS

M.K. Singal and A.R. Singal [13] initiated the study of almost continuous single-valued function between topological spaces. The concept was generalized to multivalued case and that too between bitopologicel spaces by us in [9]. It is the purpose of this section to correlate the concept with that of weakly continuous multifunction studied in the last section.

Definition 3.1 [9] For a multifunction $F : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ we define as follows.

(i) F is Q_i (P_i Q_j)-upper almost continuous in the sense of Singal and Singal (abbreviated as Q_i (P_i Q_j)-S.u.a.c.) if for each x_0 of X and each Q_i-openset V with $F(x_0) \subset V$, there exists a P_i-open nbd. of x_0 such that $F(x) \subset Q_i$-int (Q_j-cl V), for all $x \in U$.

(ii) F is called Q_i (P_i Q_j)-lower almost continuous in the sense of Singal and Singal (abbreviated as Q_i (P_i Q_j)-S.l.a.c.) if for each x_0 of X and each Q_i-open set V with $F(x_0) \cap V \neq \emptyset$, there is a P_i-open set U containing x_0 such that $F(x) \cap [Q_i$-int (Q_j-cl V) $] \neq \emptyset$, for all $x \in U$.

(iii) F is called pairwise S l.a.c. (S.u.a.c.) if F is Q_1 (P_1 Q_2)-S.l.a.c. (S.u.a.c) as well as Q_2 (P_2 Q_1)-S. l.a.c. (S u.a.c.).

(iv) F is called pairwise S. almost continuous if F is pairwise S.l.a.c and pairwise S.u.a.c.

It is obvious that

Theorem 3.2 If $F : X \to Y$ is a multifunction, then

(a) F is Q_i (P_i Q_j)-S.u.a.c. (pairwise S.u.a.c.)
$\Rightarrow$ F is Q_i (P_i Q_j)-u.w.c. (Pairwise u.w.c.)

(b) F is Q_i (P_i Q_j)-S.l.a.c. (pairwise S.l.a.c.)
$\Rightarrow$ F is Q_i (P_i Q_j)-l.w c. (pairwise l.w.c)

(c) F is pairwise S. almost continuous $\Rightarrow$ F is pairwise weakly continuous.

Definition 3.3 Let $F : X \to Y$ be a multifunction.

(a) F is called P_i (Q_i)-open, if for each P_i-open set U, F (U) is Q_i-open ($i=1$ or 2), F is called pairwise open if it is P_1 (Q_1)-open and P_2 (Q_2)-open.

(b) F is called P_i (Q_j)-point open if for each P_i-open set U, F (x) is Q_j-open, for all $x \in U$ ($i, j = 1, 2$; $i \neq j$). F is called pairwise point open if it is P_1 (Q_2)-as well as P_2 (Q_1)-point open.

Theorem 3.4 If a multifunction $F : X \to Y$ is Q_i $(P_i\ Q_j)$-u.w.c. and P_i (Q_i)-open, then F is Q_i $(P_i\ Q_j)$-S,u.a.c.

Proof. Let $x_0 \in X$ be taken arbitrarily and let V be a Q_i-open set such that $F(x_0) \subset V$. Then there exists a P_i-open nbd. U of x_0 ,such that $F(U) \subset Q_j$-cl V. Since F is P_i (Q_i)-open, $F(U) \subset Q_i$-int $(Q_j$cl V) and hence F is Q_i $(P_i\ Q_j)$-S.u.a.c.

Corollary 3.5 For a pairwise open multifunction. the concepts of pairwise S. upper almost continuity and pairwise upper weak continuity coincide.

Theorem 3.6 If a multifunction $F : X \to Y$ is Q_i $(P_i\ Q_j)$-l.w.c. and P_i (Q_j)-point open, then F is Q_i $(P_i\ Q_j)$-S.l.a.c.

Proof. Let $x_0 \in X$ be arbitrary and $V \in Q_i$ such that $F(x_0) \cap V \neq \emptyset$. Then there is a P_i-open nbd. U of x_0 such that $F(x) \cap Q_j$-cl $V \neq \emptyset$, for all $x \in U$. Now for each $x \in U$, since F (x) is Q_j-open, we must have $F(x) \cap Q_i$-int $(Q_j$-cl V) $\neq \emptyset$. for all $x \in U$. In fact, if for some $x \in U$, $F(x) \cap Q_i$-int $(Q_j$-cl V)$=\emptyset$, then since $F(x) \cap Q_j$-cl $V \neq \emptyset$ there must exist $y \in F(x)$ such that $y \in Q_j$-cl V but $y \notin Q_i$-int $(Q_j$-cl V). Then F (x) is a Q_j-open nbd. of y and hence $F(x) \cap V \neq \emptyset$, so that $F(x) \cap Q_i$-int $(Q_j$cl V) $\neq \phi$, as $V \subset Q_i$-int $(Q_j$-cl V)-a contradiction.

Hence F is Q_i $(P_i\ Q_j)$-S.l.a.c.

Corollary 3.7 For a pairwise point-open multifunction $F : X \to Y$, the concepts of pairwise S.l.a.c. and pairwise l.w.c. coincide.

The following theorem gives alternative conditions under which pairwise upper and lower weekly continuous muitifunctions may be identical with pairwise S. upper and S. lower almost continuous functions respectively.

Theorem 3.8 Let F be a multifunction from X to a pairwise regular space Y. Then.

(i) F is pairwise l.w,c. iff F is pairwise S.l.a.c.

(ii) F is pairwise u.w.c. iff F is pairwise S u.a.c., provided F (x) is pairwise strictly paracompact for each x of X.

Proof, Follows from Corollaries 2.14 and 2.16, and the fact that every pairwise lower (upper) semi-continuous function is pairwise S.l.a.c. (S.u.a.c.)

4. WEAKLY CONTINUOUS AND H. ALMOST CONTINUOUS MULTIFUNCTIONS

A new class of non-continuous single valued maps under the terminology 'almost continuous functions' was introduced by T. Husain [3]. Husain's almost continuity is seen to be independent of that of Singal and Singal [13]. This section is devoted to the introduction of an extended form of Husain's almost continuity for multifunctions in a more generalized setting of bitopological spaces. Such a multifunction is characterized and studied briefly and its behaviour with regard to weakly continuous and S. almost continuous multifunctions has been investigated.

Definition 4.1 For a multifunction $F: X \to Y$ we define as follows.

(i) F is Q_i $(P_i\ P_j)$-upper almost continuous in the sense of Husain (Q_i $(P_i\ P_j)$-H.u.a.c. - in short) if for each x_0 of X and each Q_i-open set V with $F(x_0) \subset V$, P_j-cl $F^+(V)$ is a P_i-nbd. of x_0.

(ii) F is Q_i $(P_i\ P_j)$-lower almost continuous in the sense of Husain (Q_i $(P_i\ P_j)$-H l.a.c. - in short) if for each x_0 of X and each Q_i-open set V with $F(x_0) \cap V \neq \emptyset$, P_j-cl $F^-(V)$ is a P_i-nbd. of x_0.

(iii) F is called pairwise H.l.a.c. (H.u.a c.) if F is Q_1 $(P_1\ P_2)$-H.l.a.c. (H u.a.c.) as well as Q_2 $(P_2\ P_1)$-H.l.a.c. (H.u.a,c). F is called pairwise H.a.c. if F is pairwise H l.a.c. and pairwise H.u.a.c.

Theorem 4.2 Let $F: X \to Y$ be a multifunction. Then

(a) F is Q_i $(P_i\ P_j)$-H.u.a.c. iff $F^+(V) \subset P_i$-int $[P_j$-cl $F^+(V)]$, for every Q_i- open set V.

(b) F is Q_i $(P_i\ P_j)$-H.l.a.c. iff $F^-(V) \subset P_i$-int $[P_j$-cl $F^-(V)]$. for every Q_i-open set V.

Proof.

(a) Let F be Q_i $(P_i\ P_j)$-H.u.a.c. and $x \in F^+(V)$. then $F(x) \subset V \in \cup_i$ Then P_j-cl $F^+(V)$ is a P_i-nbd. of x_0 so that $x \in P_i$-int (P_j-cl$F^+(V)$). Conversely, for any x_0 of X and a Q_i-open set V with $F(x_0) \subset V$, we have $x_0 \in F^+(V) \subset P_i$-int ($P_j$-cl $F^+(V)$).
Thus P_j-cl $F^+(V)$ is a P_i-nbd of x_0.

(b) Let F be Q_i $(P_i\ P_j)$-H.l.a.c. and $x \in F^-(V)$. Then $F(x) \cap V \neq \emptyset$. Then there exists P_i-open set U such that $x \in U \subset P_j$-cl $F^-(V)$ and hence $x \in P_i$-int $[P_j$-cl $F^-(V)]$.
Conversely, $x_0 \in X$ and $F(x_0) \cap V \neq \emptyset$, where $V \in Q_i \Rightarrow x_0 \in F^-(V) \subset P_i$ int $[P_j$-cl $F^-(V)] \subset P_j$-cl $F^-(V)$.

Theorem 4.3 A multifunction $F: (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is Q_i $(P_i\ P_j)$-H.u a c. (H l.a.c.) if the graph function $G_F: X \to (X \times Y, R_1, R_2)$ is R_i $(P_i\ P_j)$-H.u.a.c (H.l.a,c.), where $R_K = P_K \times Q_K$ (for $K = 1, 2$).

Proof. First let G_F be R_i $(P_i\ P_j)$-H.u.a.c. and let $x \in X$ be arbitrary. Suppose $V \in Q_i$ such that $F(x) \subset V$. Now $X \times V \in R_i$ such that $G_F(x) \subset X \times V$. Then P_j-cl $[G_F^+(X \times V)]$ is a P_i-nbd. of x. Now, $G_F^+(X \times V) = \{x \in X : F(x) \subset V\} = F^+(V)$. Thus P_j-cl $(F^+(V))$ is a P_i-nbd. of x and hence F is Q_i $(P_i\ P_j)$-H.u.a.c.

Next, let G_F be R_i $(P_i\ P_j)$-H.l.a.c. and $x \in X$ be arbitrary. If V is any Q_i-open set with $F(x) \cap V \neq \emptyset$, then $G_F(x) \cap (X \times V) \neq \emptyset$, where $X \times V \in R_i$. Hence there exists a P_i-open nbd. U of x such that $x \in U \subset P_j$-cl $[G_F^-(X \times V)]$. But $G_F^-(X \times V) = \{x \in X : F(x) \cap V \neq \emptyset\} = F^-(V)$ and then F is Q_i $(P_i\ P_j)$-H.l.a.c.

Corollary 4.4 A multifunction F is pairwise H.u.a.c. or pairwise H.l.a.c. or pai wise H. almost continuous if its graph function G_F is respectively so.

It can be easily seen that weak continuity and H. almost continuity of a multifunction between bitopological spaces are independent notions. In fact, they are also so even for single-valued case. We shall now derive conditions under which they can be correlated.

Definition 4.5 A multifunction $F : X \to Y$ is called

(i) Q_i $(P_j\ Q_j)$-upper almost open (**u**.a.o., in short) if
$F^+(Q_j\text{-cl}V) \subset (P_j\text{-cl}(F^+(G)))$. for every $V \in Q_i$,

(ii) Q_i $(P_j\ Q_j)$-lower almost open (l.a.o., in short) if
$F^-(Q_j\text{-cl}V) \subset P_j\text{-cl}(F^-(V))$, for every $V \in Q_i$,

(iii) pairwise u.a.o. (l.a.o.) if F is Q_1 $(P_2\ Q_2)$-u.a.o. (l.a.o.) and Q_2 $(P_1\ Q_1)$-u.a.o. (l.a.o.).

(iv) pairwise almost open if it is pairwise u.a.o. as well as pairwise l.a.o.

Theorem 4.6 A multifunction $F : X \to Y$, which is Q_i (P_iQ_j)-u.w.c. and Q_i (P_jQ_j)-u,a.o., is Q_i (P_iP_j)-H.u.a.c.

Proof. Let $x \in X$ be arbitrary and $V \in Q_i$ such that $F(x) \subset V$. Since F is Q_i (P_iQ_j)-u.w.c, there exists a P_i-open nbd U of x such that $F(U) \subset Q_j$-cl V. It then follows that $U \subset F^+(Q_j\text{-cl}V) \subset P_j\text{-cl}(F^+(V))$ (since F is Q_i (P_jQ_j)-u.a.o.). Thus P_j-cl $(F^+(V))$ is a P_i-nbd. of x and F is Q_i (P_iP_j)-H.u.a c.

Corollary 4.7. A pairwise u,a.o. multifunction is pairwise H.u.a.c. if it is pairwise u.w.c.

Theorem 4.8. A multifunction $F : X \to Y$ which is Q_i (P_iQ_j)-l.w.c. and Q_i (P_jQ_j)-l.a.o., is Q_i (P_iP_j)-H.l.a c.

Proof. Let $x_0 \in X$ and $V \in Q_i$ such that $F(x_0) \cap V \neq \emptyset$. Since F is Q_i (P_iQ_j)-l.w.c., there exists a P_i-open nbd. U of x_0 such that $F(x) \cap Q_j\text{-cl } V \neq \emptyset$, for all $x \in U$. Thus $U \subset F^-(Q_j\text{-cl } V)$. Again, since F is Q_i (P_jQ_j)-l.a.o., we have $x_0 \in U \subset F^-(Q_j\text{-cl } V) \subset P_j\text{-cl } (F^-(V))$. Thus $P_j\text{-cl } (F^-_j(V))$ is a P_i-nbd. of x_0 and F is Q_i (P_iP_j)-H.l a.c.

Coroliary 4.9- A pairwise l.a.o. multifunction is pairwise H.l.a.c. if it is pairwise l.w.c.

From Corollaries 4.7 and 4.9 we immediately have.

Corollary 4.10 A pairwise almost open and pairwise weakly continuous multifunction is pairwise H. almost continuous.

Theorem 4.11 A Q_i (P_iP_j)-H.u.a.c. multifunction $F : X \rightarrow Y$ is Q_i (P_iQ_j)-u.w.c. if $P_j\text{-cl } [F^+(V)] \subset F^+(Q_j\text{-cl } V)$, for every Q_i-open set V of Y.

Proof. Let $x \in X$ and $V \in Q_i$ such that $F(x) \subset V$. Since F is Q_i (P_iP_j)-H.u.a.c., $P_j\text{-cl } (F^+(V))$ is a P_i-nbd. of x. Then there is a P_i-open set U such that $x \in U \subset P_j\text{-cl } (F^+(V)) \subset F^+(Q_j\text{-cl } V)$. Thus $F(U) \subset Q_j\text{-cl } V$ and F is Q_i (P_iQ_j)-u.w.c.

Theorem 4.12 A Q_i (P_iP_j)-H.la.c. multifunction $F : X \rightarrow Y$ is Q_i (P_iQ_j)-l.w.c. if $P_j\text{-cl } [F^-(V)] \subset F^-(Q_j\text{-cl } V)$, for every Q_i-open set V of Y.

Proof. Let $x_0 \in X$ and $V \in Q_i$ such that $F(x_0) \cap V \neq \emptyset$. Since F is Q_i $(P_i P_j)$-H.l.a.c., there exists a P_i-open nbd. U of x_0 such that $U \subset P_j\text{-cl } (F^-(V))$. Using the given condition we have $x_0 \in U \subset F^-(Q_j\text{-clv})$. Thus $F(x) \cap Q_j\text{-cl } V \neq \emptyset$, for all $x \in U$ and F is Q_i $(P_i Q_j)$-l.w.c.

From Theorems 4.11 and 4.12 we now have

Corollary 4.13 (a) A pairwise H.u.a.c. (H,l a.c.) multifunction $F : X \rightarrow Y$ is pairwise u.w.c. (l.w.c.) if $P_j\text{-cl } [F^+(V)] \subset F^+(Q_j\text{-cl } V)$ (respectively $P_j\text{-cl } [F^-(V)] \subset F^-(Q_j\text{-cl } V)$), for every Q_i-open set V of Y, where i, j = 1 and 2, $i \neq j$.

(b) A pairwise H. almost continuous multifunction $F : X \rightarrow Y$ is pairwise weakly continuous if $P_j\text{-cl } (F^-(V)) \subset F^+(Q_j\text{-cl } V)$ holds for every Q_i-open set V of Y, where i, j = 1 and 2, $i \neq j$

The following theorems, obtained as immediate consequences of certain previously deduced results, present connections between S. almost continuity and H. almost continuity.

Theorem 4.14 (a) A multifunction $F : X \rightarrow Y$ which is Q_i $(P_i Q_j)$-S.u.a.c. (S.l.a.c.) and Q_i $(P_j Q_j)$-u.a.o. (l.a.o.) is Q_i $(P_i P_j)$-H.u.a.c. (H.l.a.c.)

(b) A pairwise u.a,o. (l.a o.) and pairwise S.u.a.c. (S l.a.c.) multifunction is pairwise H.u.a.c. (H.l.a.c.)

(c) A pairwise almost open and paitwise S. almost continuous function is pairwise H. almost continuous.

Froof. (a) Follows from Theorems 3.2, 4.6 and 4.8

(b) Follows from Theorem 3.2 and Corollary 4.7 and 4.9.

(c) Follows from Theorem 3.2 and Corollary 4.10.

Theorem 4.15 (a) A P_i (Q_i)-open and Q_i $(P_i\ P_j)$-H.u.a.c. multifunction F ; X→Y is Q_i $(P_i\ Q_j)$-S.u.a.c. if P_j-cl $[F^+(V)] \subset F^+(Q_j\text{-cl } V)$, for every O_i-open set V of Y.

(b) A pairwise open. pairwise H.u.a.c. multifunction F : X→Y is pairwise S.u.a.c. provided P_j-cl $[F^+[V)] \subset F^+(Q_j\text{-cl } V)$, for every Q_i-open set V of Y, where i, j=1 and 2, i ≠ j.

Proof. (a) Follows from Theorem 3.4 and 4.11.

(b) A consequence of Corollary 3.5 and 4.13 (a).

Theorem 4.16 (a) A P_i (Q_j)-point open and Q_i $(P_i\ P_j)$-H.l.a.c. multifunction F : X→Y is Q_i $(P_i\ Q_j)$-S.l.a.c. if P_j-cl $[F^-(V)] \subset F^-(Q_j\text{-Cl } V)$, for every Q_i-open set V of Y.

(b) A pairwise point open, pairwise H.l.a.c. multifunction F : X→Y is pairwise S.l.a.c. provided P_j-cl $[F^-(V)] \subset F^-(Q_j\text{-cl } V)$, for every Q_i-open set V of Y, where i, j=1 and 2, i ≠ j.

Proof. (a) is a direct consequence of Theorem 3.6 and 4.12, whereas (b) follows immediately from Corollary 3.7 and 4.13 (b).

Acknowledgement

The authors are thankful to the Uneversity Grants Commission for sponsoring this work under the scheme for minor research project in science.

REFERENCES

1. Azad, K. K. On fuzzy semicontinuity, fuzzy almost continuity and fuzzy Weakly continuity, Math. Analysis and Applications 82, No. 1 (1981), 14-32.

2. Frolik, Z. Remarks concerning the invariance of Baire speces under mappings, Cech, Math. J. 11 (86) (1961), 381-385.

3. Husain, T. Almost continuous mappings, Prace Mat. 10 (17) (1966), 1-7.

4. Kelly, J. C. Bitopological spaces, Proc. London Math. Soc. 13 (3) (1963), 71–89.

5. Levine, N. A decomposition cf continuity in topological spaces, Amer. Math. Monthly 68 (1961), 44-46.

6. Long, P. E. and Carnahan. D. A. Comparing almost continuous functions. Proc. Amet, Math soc. 38 (No. 2) (1973). 413–418.

7. Long, P. E, and Megehee, Jr, E. E. Properties of almost continuous functions, Proc. Amer. Math. Soc. 24 (1970), 175-180.

8. Mathur, A. and Deb, M. A note on almost continuous mappings, Math. Student 41 (No. 2) (1972), 173-184.

9. Mukherjee, M. N, and Ganguly, S. Generalization of almost continuous multifunctions to bitopological spacea, Bull. Cal. Math. Soc. 79 (1987), 274-283.

10. Noiri, T. On weakly continuous mappings, Proc. Amer, Math. Soc, 46 (No. 1) (1974) 120-124.

11. Noiri, T. Bet ween continuity and weak continuity, Boll. Un. Mat. Ital. (4) 9 (1974), 647-654.

12. Noiri. T. Semi-continuity and weak continuity, Czechoslovak Math. Jour. 31 (106) (1981), 314-321.

13. Singal. M. K. and Singal, A. R. Almost continuous mappings, Yokohama Math. Jour. 16 (No. 2) (1968), 63-73.

14. Smithson, R, E. Almost and weak continuity for multifunctions, Bull. Cal. Math. Soc. 70 (1978), 383-390.

15. Stallings, J. Fixed point theorems for connectivity maps, Funda, Math, 47 (1959), 249 263.

Received
20. 5. 1985

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